NUMERICAL SOLUTION OF INTEGRO-DIFFERENTIAL EQUATIONS USING REPRODUCING KERNEL HILBERT SPACE METHOD

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Dedication

To my parents, all members of my family and my friends.

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Mohammed H. AL-Smadi

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1	$\mathcal{H}$	Hilbert space
2	$W_2^m[a,b]$	Sobolev Space
3	L^m	Lebesgue space
4	$\{\psi_i\}_{i=1}^{\infty}$	Orthogonal function system
5	$\{\overline{\psi}_i\}_{i=1}^{\infty}$	Orthonormal function system
6	β_{ik}	Orthogonalization coefficients
7	$K_x(y)$	Reproducing kernel function
8	$R_x\left(y\right)$	Reproducing kernel function
9	Lu	Linear bounded operator
10	$u\left(x\right)$	Analytical Solution
11	$u_n\left(x\right)$	Approximate solution
12	BVP	Boundary Value Problem
13	IVP	Initial Value Problem
14	RKHS	Reproducing Kernel Hilbert Space
15	PDE	Partial Differential Equation
16	IDE	Integro-Differential Equation
17	BC's	Boundary Conditions
18	Abs.C	Absolutely continuous

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Abstract

In this thesis, we make a discussion for the theory of reproducing kernel Hilbert spaces associated with positive definite kernels and we illustrate their applications for a class of integro differential equations.

We begin with the material that is contained in Aronszajn's classic paper on the theory of reproducing kernel Hilbert spaces. In fact, we focus on their properties, generation of new spaces and relationships between their kernels and some theorems on extensions of functions and kernels. Moreover, we study the Sobolev space which is one of the most useful reproducing kernel Hilbert spaces, construct a novel reproducing kernel space and give the way to express reproducing kernel functions. Meanwhile, we employed a reproducing kernel function and its conjugate operator to construct the complete orthonormal basis in the space $W_2^m[a, b]$.

This thesis investigates the solutions of a general form of first, second and fourth-order integro-differential equations using the reproducing kernel Hilbert space method. The analytical and approximate solution are represented in the form of series in the space $W_2^m[a, b]$. The n-term approximation and all its derivatives are obtained and are

proved to converge uniformly to the analytical solution and all its derivatives, respectively.

Furthermore, our method has the following advantages: small computational requirements, high precision and it is possible to pick any point in the interval of integration and as well the approximate solutions and its all derivatives will be applicable.

The experimental results show that only a few number of iteration steps can be used to attain a high degree of accuracy. So, various numerical examples are presented to illustrate the computational efficiency, the reliability, and the accuracy of the proposed method for integro-differential equations of first, second and fourth-order.

Key Words: Integro-Differential Equation, Positive Definite Functions, Sobolev Spaces, Reproducing Kernel Theory, Reproducing Kernel Method.

Introduction

The reproducing kernel was used for the first time at the beginning of the 20^{th} century by S. Zaremba in his work on boundary value problems (BVPs) for harmonic and biharmonic functions. In 1907, he was the first who introduced, in a particular case, the kernel corresponding to a class of functions, and stated its reproducing property. But he did not develop any theory and did not give any particular name to the kernels he introduced.

In 1909, J. Mercer examined the functions which satisfy reproducing property in the theory of integral equations developed by Hilbert. The kernels considered then were continuous kernels of positive definite integral operators, this theory was developed by him under the name of "*positive definite kernels*". Furthermore, he showed that these positive definite kernels have nice properties among all continuous kernels of integral equations.

In the 1920s, the idea of reproducing kernels appeared in the dissertations of three Berlin mathematicians G. Szegö (1921), S. Bergman (1922) and S. Bochner (1922). In particular, S. Bergman introduced reproducing kernels in one and several variables for the class of harmonic and analytic functions and he called them "*kernel functions*".

However, E. H. Moore (1935) introduced the positive definite kernels in the general analysis under the name of "*positive Hermitian matrices*" with a view to applications in a kind of generalization of integral equations. In 1943, N. Aronszajn developed the theory of reproducing kernels which contains the Bergman kernel functions.

The original idea of Zaremba to apply the kernels to the solution of BVPs was developed by S. Bergman and M. Schiffer (1947). In these investigations, the kernels were proved to be powerful tool for solving BVPs of partial differential equations (PDEs) of elliptic type. Moreover, by application of kernels to conformal mapping of multiply-connected domains, very beautiful results were obtained by S. Bergman and M. Schiffer (1948).

The general theory of reproducing kernels was systematized by N. Aronszajn (1950). He summarized the previous related research results and used "*reproducing kernel functions*" as the identical term for these different functions, so the foundations of reproducing kernel theory was set up.

The basic result by Aronszajn is the existence and uniqueness of a reproducing kernel Hilbert space (RKHS) corresponding to any *self-adjoint nonnegative-definite kernel* $K(p,q)$, $p,q \in E$, where E is a nonempty abstract set. Let $\mathcal{H}$ be a Hilbert space of functions defined on E, and $\mathcal{H} \subset L^2(E)$. Assume that $K(\cdot,q)$ and $K(p,\cdot)$ belong to $\mathcal{H}$. Let us assume that the linear operator $K : \mathcal{H} \longrightarrow \mathcal{H}$, with the kernel $K(p,q)$, is injective. It is defined on all of $\mathcal{H}$ since $K(p,\cdot) \in \mathcal{H}$ by the assumption. Define RKHS $\mathcal{H}_K$ inner product by the formula

$$(f,g)_{\mathcal{H}_K} := \langle f,g \rangle := \left\langle K^{-1}f, g \right\rangle = \langle\, K^{-\frac{1}{2}}f, K^{-\frac{1}{2}}g \,\rangle\,, \tag{0.1}$$

where $\langle f,g\rangle := \langle f,g\rangle_{L_2(E)}$, K^{-1} is the operator inverse to $K : \mathcal{H} \longrightarrow \mathcal{H}$, and $Kf := \int_E K(p,q)\; f(q)dq$.

The injectivity assumption can be dropped, but then one has to consider K on the factor space $\mathcal{H}/N(K)$, where $N(K) := \{f : Kf = 0\}$ is the null-space of K. The reproducing property of the kernel $K(p,q)$ can be stated as follows:

$$\langle f(\cdot), K(\cdot,q)\rangle = f(q), \tag{0.2}$$

and this formula can be easily derived from the definition (0.1) of the inner product in $\mathcal{H}_K$ as $\langle f(\cdot), K(\cdot,q)\rangle := (K^{-1}f, K(\cdot,q)) = (f, K^{-1}K(\cdot,q)) = (f, I(\cdot,q)) = f(q)$. Here, we have used the self-adjointness of the operator K^{-1} and the fact that the distributional kernel of the identity operator I is $\delta(p-q)$, the delta function, which is well defined on RKHS because the value $f(p)$ for any $p \in E$ is a bounded linear functional in $\mathcal{H}$: $|f(q)| \leq \|f\|\,\|K(\cdot,q)\|$, where $\|f\| := \langle f,f\rangle^{\frac{1}{2}}$ is the norm in $\mathcal{H}_K$.

In brief, a function $K : E \times E \to \mathbb{F}$, where $\mathbb{F}$ is a field, is a reproducing kernel of the Hilbert space $\mathcal{H}$ if for each $t \in E$ and $f \in \mathcal{H}$, we have $K(\cdot, t) \in \mathcal{H}$ and $\langle f, K(\cdot, t)\rangle = f(t)$. The last condition is called "*the reproducing property*": the value of the function f at the point t is reproducing by the inner product of f with $K(\cdot, t)$.

A Hilbert space that is endowed with such a kernel is called "*a reproducing kernel Hilbert space*" or "*a proper Hilbert space*". Conversely, a function on $E \times E$ for which such a Hilbert space $\mathcal{H}$ exists is a reproducing kernel and we usually write $\mathcal{H}_K$ for this space which is unique.

Since then, many researchers have contributed much work to the development and improvement of reproducing kernel theory.

In 1963, L. Schwartz showed that Mercer's theorem also holds true for general compact spaces. Moreover, the use of Mercer's theorem for interpreting kernels as inner products in a feature space was introduced by Aizerman, Braverman and Rozonoer (1964).

M.G. Cui (1986) proved that the space $W_2^1[a, b]$ is a Hilbert space with reproducing kernel function, and exactly expressed the reproducing kernel function of the space $W_2^1[a, b]$ by finite terms. Hence, the application of reproducing kernel theory began in many areas.

In the 1990s, the emergence of support vector machines, starting with G. Wahba (1990), E. Boser (1992), has elucidated the connection between splines and reproducing kernel methods (RKMs).

Based on the reproducing kernel space $W_2^m[a, b]$ established by M.G. Cui (1998) and by re-defining the inner product, the reproducing kernel functions of the space $W_2^m[a, b]$ can be significantly simplified and represented by piecewise polynomials and the higher order of derivatives without changing any other conditions. Such expressions of reproducing kernel functions are the simplest from the computational viewpoint, the speed and accuracy could be significantly improved in scientific and engineering applications.

M.G. Cui and C.L. Li (2003) introduced the Sobolev space $W_2^1(\Omega^n)$ with the reproducing properties and discussed the structure of the solution space of operator equations. Thereafter, F. Geng and M.G. Cui presented many research reports based on the theory of numerical analysis in the RKHS.

Second-order BVPs of integro-differential equations (IDEs) was solved by W. Yulan (Yulan, et al., 2009) using the RKHS. Meanwhile, Y. Zhou and Y. Lin presented a simple method for solving the Fredholm singular IDEs with Cauchy kernel, using a transformation and modifying the traditional RKMs.

In recent years, iterative RKMs with global convergence was applied to obtain the analytical approximate solution for many problems. For instance, singular initial value problems (IVPs), regular, singular, singular weakly, singular periodic, and singularly perturbed BVPs, system of regular and singular IVPs, and BVPs, regular and singular integral equations (IEs), PDEs and inverse problems in PDEs.

Our work is mostly based on F. Geng's papers and lecture notes (Geng, et al., 2007, a & b, 2009, a, b & c, 2010 & 2011, a & b) in its diversity of tools and results, and N. Aronszajn's works (Aronszajn, 1950). Moreover, we used partially the papers of M.G. Cui (Cui, et al., 2003, 2006, a & b, 2007, a, b & c, 2009, 2010 & 2011, a & b) and (Alpay, 2003 & 2005) for complementing with our results.

The thesis is organized as follows: In Chapter One, we give a brief introduction about RKHS; preliminaries, basic concepts, definitions and theorems relevant to our study. Afterward, we present the reproducing kernel function by re-defining the inner product of a reproducing kernel space in order to obtain the analytical approximate solution for a general form of IDEs. Furthermore, the analysis of the RKHS method is described and an effective algorithm is introduced.

In Chapter Two, the RKHS method is applied to approximate the solution of a general form of first-order IDEs. It is a relatively new analytical technique. The analytical solution $u(x)$ and approximate solution $u_n(x)$ are represented in the form of series in the space $W_2^2[a,b]$. Meanwhile, we give an iterative method to solve a nonlinear first-

order IDEs. Various numerical examples are presented to illustrate the computational efficiency and the accuracy of the proposed method.

In Chapter Three, we apply the RKHS to develop a novel numerical method in the space $W_2^3[a,b]$ for obtaining approximations to the solution and its derivatives for a general form of second-order Volterra IDEs. It is a relatively new analytical technique. The analytical solution $u(x)$ and approximate solution $u_n(x)$ are represented in terms of series in the space $W_2^3[a,b]$. The n-term approximation is obtained and is proved to converge to the analytical solution. Moreover, two numerical examples are studied to demonstrate the accuracy of the present method.

RKHS method is applied to approximate the solution of two-point BVPs for fourth-order IDEs in Chapter Four. The analytical and the approximate solution are represented in the form of series in the space $W_2^5[a,b]$. Moreover, the approximate solution and all its derivatives converge uniformly to the exact solution and all its derivatives, respectively. The experimental results show that only a few number of iteration steps can be used for numerical purpose with a high degree of accuracy. This thesis ends in Chapter Five with some concluding remarks and future recommendations.

Chapter One

Basic Fundamentals

This chapter contains six sections. In section one, we will present the basic concepts, definitions and theorems within the scope necessary for this thesis. In section two, we will review the standard function spaces: The spaces of differentiable functions, the Lebesgue spaces and the Sobolev spaces. Since kernels are of particular importance, we will study them in more detail in section three. We will therefore discuss positive definite functions and their properties. Afterward, we will introduce a RKHS with the aim of constructing such a feature space.

In section four, we will find out the expression forms of reproducing kernel functions in the space $W_2^m[a,b]$. These expressions can be represented by piecewise polynomial of degree $2m-1$. It is worthwhile to point out that the reproducing kernel function has a unique representation. Further, we will give some corollaries and remarks related to these kernel functions. At the end of this section, several examples of such kernel functions are given in the space $W_2^1[0,1]$.

In section five, we will establish an iterative method to construct and calculate the reproducing kernel for the general m^{th}-order BVP. The formula for calculation and the representation of the exact solution is given in the reproducing kernel space. Its approximate solution is obtained by truncating the n-term of the exact solution. Based on the RKHS method, we will introduce an effective algorithm for nonlinear IDE in the last section.

1.1 Preliminaries

We will now introduce the notations and symbols used in this thesis and give a brief overview over the function standard spaces of interest. The symbols $\mathbb{R}$ and $\mathbb{C}$ indicate the set of real and complex numbers. We will use $\mathbb{F}$ to denote either $\mathbb{R}$ or $\mathbb{C}$, so that

when we wish to state a definition or result that is true for either the real or complex numbers, we will use $\mathbb{F}$.

A vector or a vector-valued function is always bold printed. The jth component of a vector $\mathbf{x}$ is denoted by x_j . The complex conjugate of a number z is $\overline{z}$ and the absolute value is denoted by $|z|$. Throughout the whole thesis, $\{\psi_i\}_{i=1}^{\infty}$ and $\{\overline{\psi}_i\}_{i=1}^{\infty}$ refer to the orthogonal and orthonormal function system, respectively.

The Greek letters α, β and γ refer to real numbers. The capital Latin letters $\mathcal{H}$ and $\mathcal{W}$ refer to RKHS and the letters K, R and T are always used for its reproducing kernel function.

A linear and continuous map φ from a Hilbert space $\mathcal{H}$ to the real numbers is called a functional, that is, φ is an element of the dual space $\mathcal{H}^*$. We denote the point evaluation functional by δ_x, where $x \in \mathcal{X}$. The functional $\delta_x(f) = f(x)$, for all $f \in \mathcal{H}$, δ is the Dirac evaluation functional.

We suppose that the reader is familiar with concepts of linear, topological, Banach and Hilbert spaces. A pre-Hilbert space "*inner product space*" $\mathcal{H}_0$ is a linear space over $\mathbb{C}$ with an inner product defined on it. If $\mathcal{H}$ is an inner product space, $\|\cdot\|$ defines a norm on $\mathcal{H}$ by $\|f\| = \langle f, f\rangle^{1/2}$. Sometimes, we use the notations $\|\cdot\|_{\mathcal{H}}$ and $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ for norm and inner product, respectively. Moreover, A pre-Hilbert space $\mathcal{H}_0$ is called a Hilbert space if it is complete in metric. In other words, if f_n is a Cauchy sequence in $\mathcal{H}_0$, *i.e.*, $\|f_m - f_n\| \to 0$, when $m, n \to \infty$, then there is $f \in \mathcal{H}_0$ such that $\|f_n - f\| \to 0$ when $n \to \infty$.

Let $\mathcal{H}_0$ be a pre-Hilbert space. We can imbed $\mathcal{H}_0$ as a dense subspace of a Hilbert space $\mathcal{H}$ in such a way that $\langle f, g\rangle_{\mathcal{H}} = \langle f, g\rangle_{\mathcal{H}_0}$ for $f, g \in \mathcal{H}_0$. This canonical method is called *completion*.

In addition, for a Hilbert function space $\mathcal{H}$, the *Schwarz inequality* holds: $|\langle f, g\rangle| \leq \|f\| \, \|g\|$ for $f, g \in \mathcal{H}$ and equality occurs when f and g are linearly dependent. Let f and g be square-integrable function. Then $\left|\int f(x)\, g(x)\, dx\right|^2 \leq \int |f(x)|^2\, dx \int |g(x)|^2\, dx$.

Define $d(f,g) := \|f-g\|$ the binary function. Observe that for f, g and $h \in \mathcal{H}$ and for all $\xi \in \mathbb{C}$, d satisfies all requirements to be a metric:

(i) $d(f,g) \geq 0$ and equality occurs if and only if $f = g$.

(ii) $d(f,g) = d(g,f)$.

(iii) $d(f,g) \leq d(f,h) + d(h,g)$.

(iv) $d(f-h, g-h) = d(f,g)$ (translation invariant).

(v) $d(\xi f, \xi g) = |\xi|\, d(f,g)$.

This binary function introduces metric topology in $\mathcal{H}$ which is called *strong topology* or *norm topology*. In other words, a sequence $(f_n)_{n\geq 0}$ in $\mathcal{H}$ converges strongly to a function f if $\|f_n - f\| \to 0$, as $n \to \infty$.

A function with m continuous derivatives is called a C^m function. $C^m\left(\overline{\Omega}\right)$ is the *Banach space* of all functions in $\overline{\Omega}$ such that $f(x)$ and $\partial^\alpha f(x)$ with $|\alpha| \leq m$, α is a multi–index, are *uniformly continuous* in $\overline{\Omega}$ and the norm

$$\|f\|_{C^m\left(\overline{\Omega}\right)} = \max_{x\in\Omega}\left\{\sum_{|\alpha|\leq m} |\partial^\alpha f(x)|\right\} \text{ is finite,}$$

where m is positive integer.

If $m = 0$, we denote $C^0\left(\overline{\Omega}\right) = C\left(\overline{\Omega}\right)$. If Ω is bounded, then $\|f\|_{C^m\left(\overline{\Omega}\right)} \leq \infty$ follows from the uniform continuity of f, $\partial^\alpha f$, $|\alpha| \leq m$. Moreover, $C^m(\Omega)$ is the class of functions in Ω such that $f(x)$ and $\partial^\alpha f(x)$, $|\alpha| \leq m$ are continuous in Ω. If $m = 0$, we denote $C^0(\Omega) = C(\Omega)$ the set of all continuous functions with the maximum norm. Note that if Ω is compact, then $C^m\left(\overline{\Omega}\right) = C^m(\Omega)$.

A functional f is said to be bounded if there exists a constant $M > 0$ such that for all $x \in \mathcal{X}$, $|f(x)| \leq M\|f\|$. Note that a bounded functional is also continuous. The Riesz representation theorem states that if $f : \mathcal{H} \longrightarrow \mathbb{F}$ is a continuous linear functional mapping from a Hilbert space $\mathcal{H}$ to a scalar field, then for all $x \in \mathcal{H}$, there exists a

unique $y_f \in \mathcal{H}$ such that $f(x) = \langle x, y_f \rangle$. In RKHS, we will study the case when $\mathcal{H}$ is a Hilbert function space.

1.2 The Spaces and Their Norms

In this section, we will present a more global notion of derivative suited to the Lebesgue space. Afterward, we will give a brief overview on basic results of the theory of Sobolev spaces.

1.2.1 Lebesgue Spaces

We refer to $L^p(\Omega)$, $1 \leq p < \infty$, as the linear space of $p-th$ order integrable functions on Ω and to $L^\infty(\Omega)$ as the linear space of essentially bounded functions which are Banach spaces with respect to the norms $\|u\|_{L^P} := \left(\int_\Omega |u(x)|^p \, dx\right)^{1/p}$ is finite, and $\|u\|_{L^\infty} := ess \ \sup_{x\in\Omega} |u(x)|$.

Note that for $p = 2$, the space $L^2(\Omega) := \{u : \int_\Omega u^2(x) \ dx < \infty\}$ is a Hilbert space with respect to the inner product $\langle u, v \rangle_{L^2} := \int_\Omega uv \ dx$.

Let E be a compact subset of Ω, the set of locally integrable functions is denoted by $L^1_{loc}(\Omega) = \{u : u \in L^1(E), \ \ \forall E \subset \Omega\}$.

Now, we will define the weak derivative which is a global notion of derivative suited to the Lebesgue space.

Let α be a multi–index and $u \in L^1_{loc}(\Omega)$. The function u is said to have a weak derivative $\partial^\alpha u$, if there exists a function $v \in L^1_{loc}(\Omega)$ such that

$$\int_\Omega u(x) \, \partial^\alpha \varphi(x) \, dx = (-1)^{|\alpha|} \int_\Omega v(x) \, \varphi(x) \, dx, \quad \varphi \in C_0^\infty(\Omega),$$

where $C_0^\infty(\Omega)$ is the space of all function in $C^\infty(\Omega)$ with compact support. If such v exists, we define $\partial^\alpha u := v$.

Hence, the classical derivative $\partial^\alpha u$ is also the weak derivative. But, $\partial^\alpha u$ may exist in the weak sense without existing in the classical sense.

The existence of the weak derivative is related to the absolute continuity property. Now, we recall the definition of absolute continuity for function of one variable.

A function $u : [a,b] \longrightarrow \mathcal{R}$ is called absolutely continuous (Abs.C), if for every positive ε there exists a positive δ such that for any finite set of disjoint intervals $(x_1, y_1), (x_2, y_2), ..., (x_k, y_k)$ $(\subset [a,b])$ with $\sum_{j=1}^{k} |y_j - x_j| < \delta$, one has

$$\sum_{j=1}^{k} |u(y_j) - u(x_j)| < \varepsilon.$$

Next, we will use the following facts:

1. A function $u : [a,b] \longrightarrow \mathcal{R}$ is Abs.C if and only if there exists a function $v \in L^1(a,b)$ such that $u(x) = u(a) + \int_a^x v(t)\,dt,\ x \in [a,b]$. Actually, this is known as *the fundamental theorem of Lebesgue integral calculus.*

2. If $u : [a,b] \longrightarrow \mathcal{R}$ is Abs.C, then there exists derivative $\frac{du}{dx}$ for almost every $x \in (a,b)$ and $\frac{du}{dx} = v$ $(\in L^1(a,b))$.

Hence, $u : [a,b] \longrightarrow \mathcal{R}$ is Abs.C on $[a,b]$ if and only if there exists the weak derivative $\frac{du}{dx} \in L^1(a,b)$. The weak derivative coincides with the classical derivative almost everywhere.

In the next, we will utilize the definition of the Sobolev Space $W_p^m(\Omega)$ which is one of the most useful RKHS.

1.2.2 Sobolev Spaces

Sobolev spaces are based on the concept of weak derivatives. Further, many Sobolev spaces can be characterised as RKHS.

Let m be non-negative integer and $u \in L^p(\Omega)$. Suppose that the weak derivatives $\partial^\alpha u$ exist for any α with $|\alpha| \leq m$ (all derivatives up to order m), such that $\partial^\alpha u \in L^p(\Omega)$, $|\alpha| \leq m$. Then we say that $u \in W_p^m(\Omega)$.

The function space $W_p^m(\Omega)$ given by $W_p^m(\Omega) = \{u \in L^p(\Omega) : \partial^\alpha u \in L^p(\Omega), |\alpha| \leq m\}$ is called a Sobolev space.

It is a Banach space with respect to the (standard) norm

$$\|u\|_{W_p^m(\Omega)} := \left(\sum_{|\alpha| \leq m} \|\partial^\alpha u\|_{L^p(\Omega)}^p \right)^{1/p} = \left(\int_\Omega \sum_{|\alpha| \leq m} |\partial^\alpha u|^p \, dx \right)^{1/p}, \; p \in [1, \infty),$$

and $\|u\|_{W_\infty^m(\Omega)} := \max_{|\alpha| \leq m} \|\partial^\alpha u\|_{L^\infty(\Omega)}, \, p = \infty.$

From above, we note the following:

1. If $m = 0$, the space $W_p^0(\Omega) = L^p(\Omega)$.

2. If $p = 2$, the space $W_2^m(\Omega)$ is a Hilbert space with the inner product

$$\langle u, v \rangle_{W_2^m} := \int_\Omega \sum_{|\alpha| \leq m} \partial^\alpha u(x) \, \partial^\alpha v(x) \, dx.$$

In particular, we'll use the m-th order Sobolev space $W_2^m[a,b]$ which is defined as follows:

$$W_2^m[a,b] = \{u : u^{(j)} \text{ is Abs.C}, \; j = 1, 2, .., m-1, \; u^{(m)} \in L^2[a,b]\} \tag{1.1}$$

The space $W_2^m[a,b] \subset L^2[a,b]$. Hence, the properties of $L^2[a,b]$ functions will be applicable to the elements of the space $W_2^m[a,b]$. The inner product and the norm in the function space $W_2^m[a,b]$ are defined as follows respectively: for any functions $u(x), v(x) \in W_2^m[a,b]$,

$$\langle u, v \rangle_{W_2^m[a,b]} := \sum_{i=0}^{m-1} u^{(i)}(a) \, v^{(i)}(a) + \int_a^b u^{(m)}(x) \, v^{(m)}(x) \, dx, \tag{1.2}$$

$$\|u\|_{W_2^m[a,b]} = \sqrt{\langle u, u \rangle_{W_2^m[a,b]}}. \tag{1.3}$$

For $W_2^m[a,b]$ another notation $\mathcal{H}^m[a,b]$ is often used: $W_2^m[a,b] = \mathcal{H}^m[a,b]$.

1.3 Reproducing Kernel Hilbert Space

Since N. Aronszajn put forward the RKHS theory in 1950s, many researchers have done much works in this field. Especially in recent 20 years, more and more experts have seen the advantages of reproducing kernel space.

A RKHS arise in a variety of pure and differential applied sciences, it was one of the main motivations for the development of the approximation theory, machine learning theory, group representation theory and various areas of complex analysis.

Now, let us start by defining a kernel, since the term "kernel" itself is used in different branches of mathematics. Our main requirement on kernels is that they are positive definite, the positive definiteness assumption is a key assumption of the functional view described later in the RKHS theory.

Let Φ be a feature mapping from $\mathcal{X}$ into the space of functions $\mathcal{H}$ so that $\Phi(x) = K(x,\cdot)$. A function $K : \mathcal{X} \times \mathcal{X} \longrightarrow \mathbb{R}$ satisfying $K(x,x') = \langle \Phi(x), \Phi(x') \rangle$ for all $x, x' \in \mathcal{X}$ is called a kernel. The *Mercer's theorem* gives a way to determine whether an arbitrary symmetric function $K : \mathcal{X} \times \mathcal{X} \longrightarrow \mathbb{R}$ is a kernel. Let $x_1, ..., x_n$ be a collection of elements of $\mathcal{X}$, and let $\mathbf{K}$ be the Gram matrix defined by $\mathbf{K}_{ij} = K(x_i, x_j)$. Then the theorem states that K can be expressed as an inner product if any Gram matrix $\mathbf{K}$ formed by any collection of elements of $\mathcal{X}$ is a positive semi-definite matrix. This is equivalent to saying that K is positive semi-definite, *i.e.*, for all f, we have $\int\int f(x) K(x.x') f(x') \, dx dx' \geq 0$.

Definition 1.3.1 (Aronszajn, 1950) *Let $\mathcal{X}$ be an arbitrary set and K be a kernel on $\mathcal{X}$, that is, $K : \mathcal{X} \times \mathcal{X} \longrightarrow \mathbb{C}$. The kernel K is called Hermitian if for any finite set of points $\{x_1, ..., x_n\} \in \mathcal{X}$ and any complex numbers $\alpha_1, ..., \alpha_n$ we have $\sum_{i,j=1}^{n} \overline{\alpha}_i \alpha_j K(x_i, x_j) \in R$.*

Definition 1.3.2 (Aronszajn, 1950) *Let $\mathcal{X}$ be an arbitrary set. A symmetric function $K : \mathcal{X} \times \mathcal{X} \longrightarrow \mathbb{C}$ is called positive definite kernel if* $\sum_{i,j=1}^{n} \overline{\alpha}_i \alpha_j K(x_i, x_j) \geq 0$, *for any $n \in \mathcal{N}$, $x_1, ..., x_n \in \mathcal{X}$ and $\alpha_i \in \mathbb{C}$.*

Equivalently, for any finitely supported family of complex numbers $\{\alpha_x\}_{x\in\mathcal{X}}$ we have $\sum_{x,y\in\mathcal{X}} \overline{\alpha}_y \alpha_x K(x, y) \geq 0$. Sometimes, we will denote this by $K(x, y) \geq 0$ on $\mathcal{X}$.

We will now introduce a reproducing kernel on a Hilbert space $\mathcal{H}$. Next, we will note that a RKHS can be constructed for every positive definite kernel. It is a Hilbert space characterized by a kernel that reproduces, via an inner product, every function of the space.

Definition 1.3.3 (Aronszajn, 1950) *Let $\mathcal{H}$ be a Hilbert space of functions $f : \mathcal{X} \longrightarrow \mathbb{F}$ on a set $\mathcal{X}$. A function $K : \mathcal{X} \times \mathcal{X} \longrightarrow \mathbb{C}$ is a reproducing kernel of $\mathcal{H}$ if the followings are satisfied:*

1. $K(\cdot, x) \in \mathcal{H}$ for all $x \in \mathcal{X}$.

2. $\langle f, K(\cdot, x)\rangle = f(x)$ for all $f \in \mathcal{H}$ and for all $x \in \mathcal{X}$.

The last condition is called "*the reproducing property*": the value of the function f at the point x is reproducing by the inner product of f with $K(\cdot, x)$. Further, one can rewrite the first condition as: for all $x \in \mathcal{X}$, $K_x(y) = K(x, y)$ as a function of y belongs to $\mathcal{H}$, $y \in \mathcal{X}$.

So applying the reproducing property to the function K_x at y, we get $K_x(y) = \langle K_x(\cdot), K_y(\cdot)\rangle$, for $x, y \in \mathcal{X}$, *that is*, $K(x, y) = \langle K_x, K_y\rangle$ for $x, y \in \mathcal{X}$. Consequently, for all $x \in \mathcal{X}$ we obtain

$$\|K_x\|^2 = \langle K_x, K_x\rangle = K(x, x). \tag{1.4}$$

A Hilbert space $\mathcal{H}$ of functions on a set $\mathcal{X}$ which admits a reproducing kernel K is called a RKHS.

We could equivalently define a RKHS as: let $\mathcal{H}$ be a Hilbert function space over $\mathcal{X}$. For an element $x \in \mathcal{X}$, a Dirac functional is a function $\delta_x \in \mathcal{H}$ such that for all $f \in \mathcal{H}$, we have $\delta_x(f) = f(x)$. Then $\mathcal{H}$ is a RKHS if all Dirac functional in $\mathcal{H}$ are bounded, that is, RKHS is a Hilbert function space with the property that every point evaluation functional is bounded and linear. For instance, the L^2 space is a Hilbert space, but not a RKHS because the delta function which has the reproducing property $f(x) = \int_\Omega \delta(x-y) f(y)\, dy$ does not satisfy the square integrable condition, that is, $\int_\Omega \delta(y)^2 dy \nless \infty$, thus the delta function is not in L^2. The Hilbert space with reproducing kernel K is denoted by $\mathcal{H}_K$. Further, $\langle \cdot, \cdot \rangle_{\mathcal{H}_K}$, $\|\cdot\|_{\mathcal{H}_K}$ denote the inner product and the norm, respectively.

Theorem 1.3.1 (Aronszajn, 1950) *If a Hilbert space $\mathcal{H}$ of functions on a set $\mathcal{X}$ admits a reproducing kernel, then the reproducing kernel $K(x,y)$ is uniquely determined by the Hilbert space $\mathcal{H}$.*

Proof. *Let $K(x,y)$ be a reproducing kernel of $\mathcal{H}$. Suppose that there exists another kernel $R(x,y)$ of $\mathcal{H}$. Then, for all $x \in \mathcal{X}$, applying the reproducing property for K and R we get $\|K_x - R_x\|^2 = \langle K_x - R_x, K_x - R_x \rangle = \langle K_x - R_x, K_x \rangle - \langle K_x - R_x, R_x \rangle = (K_x - R_x)(x) - (K_x - R_x)(x) = 0$*

Hence $K_x = R_x$, that is, $K_x(y) = R_x(y)$ for all $y \in \mathcal{X}$. This means that $K(x,y) = R(x,y)$ for all $x, y \in \mathcal{X}$. The proof is complete.

Conversely, if a symmetric positive definite kernel is given, then one can construct a unique RKHS in which the given kernel acts as the reproducing kernel. The construction of the RKHS from a given positive definite kernel $K(x,y)$ is achieved as follows. We defined a pre-Hilbert space $H_0(\mathcal{X})$ by

$$\operatorname{span}\{K(\cdot,x) : x \in \mathcal{X}\} := \{f(\cdot) = \sum_{i=1}^{n} \alpha_i K(\cdot, x_i) : x_i \in \mathcal{X}, \alpha_i \in \mathcal{F}\}.$$

The inner product between $f = \sum_{i=1}^{n} \alpha_i K(\cdot, u_i)$ and another function $g = \sum_{j=1}^{m} \beta_j K(\cdot, v_j)$

in this space is defined as $\langle f, g\rangle_{H_0} := \sum_{i=1}^{n}\sum_{j=1}^{m}\alpha_i\overline{\beta_j}K(u_i, v_j)$.

An interesting property that follows directly from the definition of K, for all $f \in H_0(\mathcal{X})$, is $\langle f, K(\cdot, x)\rangle_{H_0} = \sum_{i=1}^{n}\alpha_i K(x, x_j) = f(x)$, *i.e.*, K has the reproducing property.

By taking a completion with respect to the norm of $H_0(\mathcal{X})$, we can define the RKHS $H_k(\mathcal{X})$ with the inner product $\langle \cdot, \cdot\rangle_{H_K} = \langle \cdot, \cdot\rangle_{H_0}$. See (Wendland, 2005) for more details of the construction.

In brief, a Hilbert space $\mathcal{H}$ of functions $f : \mathcal{X} \longrightarrow \mathbb{C}$ is RKHS if $\exists$ positive definite kernel $K(x, y)$, *s.t.*

1. K has the reproducing property, *i.e.*, $\langle f(\cdot), K(\cdot, x)\rangle = f(x)$, $\forall f \in \mathcal{H}, \forall x \in \mathcal{X}$, and in particular, $\langle K_x(\cdot), K_y(\cdot)\rangle = K_x(y)$.

2. K spans $\mathcal{H}$, *i.e.*, every $f \in \mathcal{H}$ can be written as $f(\cdot) = \sum \alpha_i K(\cdot, x_i)$.

Theorem 1.3.2 *For a Hilbert space $\mathcal{H}$ of functions on $\mathcal{X}$, there exists a reproducing kernel K for $\mathcal{H}$ if and only if for every x of $\mathcal{X}$, the evaluation linear functional $\mathcal{I} : f \longrightarrow f(x)$ is a bounded linear functional on $\mathcal{H}$.*

Proof. *See (Aronszajn, 1950).*

Theorem 1.3.3 *The reproducing kernel $K(x, y)$ of a RKHS H is a positive definite kernel.*

Proof. *See (Aronszajn, 1950).*

Next, we will use the following remarks:

Remark 1.3.1 *Let $\mathcal{H}$ be a RKHS and its kernel $K(x, y)$ on $\mathcal{X}$, then, for all $x, y \in \mathcal{X}$, we have*

(i) $K(y, y) \geq 0$.

(ii) $K(x, y) = \overline{K(y, x)}$.

(iii) $|K(x,y)|^2 \leq K(x,x)\, K(y,y)$, (Schwarz Inequality).

(iv) Let $x_0 \in X$. Then the followings are equivalent:

(a) $K(x_0, x_0) = 0$.

(b) $K(x_0, y) = 0$ for all $y \in \mathcal{X}$.

(c) $f(x_0) = 0$ for all $f \in \mathcal{H}$.

Indeed, (i) and (ii) can be easily seen. For (iii), we use the Schwarz Inequality in $\mathcal{H}$ and get

$$\begin{aligned} |K(x,y)|^2 &= |\langle K_x, K_y\rangle|^2 \leq (\|K_x\|\,\|K_y\|)^2 = \|K_x\|^2\,\|K_y\|^2 \\ &= \langle K_x, K_x\rangle \langle K_y, K_y\rangle = K(x,x)\, K(y,y). \end{aligned}$$

For (iv), it follows by (iii) that $|K(x_0,y)|^2 \leq K(x_0,x_0)\, K(y,y) = 0$. Hence $K(x_0,x_0) = 0$ is equivalent with $K(x_0,y) = 0$ for all $y \in \mathcal{X}$. Further, by the reproducing property, $K(x_0,y) = 0$ for all $y \in \mathcal{X}$ if and only if $f(x_0) = 0$, for all f.

Theorem 1.3.4 *Every sequence of functions $(f_n)_{n\geq 1}$ which converges strongly to a function f in $H_k(\mathcal{X})$, converges also in the pointwise sense, that is, $\lim_{n\to\infty} f_n(x) = f(x)$, for any point $x \in \mathcal{X}$. Further, this convergence is uniform on every subset of $\mathcal{X}$ on which $x \mapsto K(x,x)$ is bounded.*

Proof. *See (Aronszajn, 1950).*

Theorem 1.3.5 *The function space $W_2^m[a,b]$ is a reproducing kernel space. That is, for each fixed $x \in [a,b]$ and any $u(y) \in W_2^m[a,b]$, there exists $K_x(y) \in W_2^m[a,b]$, $y \in [a,b]$ such that $\langle u(y), K_x(y)\rangle = u(x)$, and $K_x(y)$ is called the reproducing kernel function of space $W_2^m[a,b]$.*

Proof. *See (Aronszajn, 1950).*

1.4 Reproducing Kernel Functions

In this section, we will find out the expression forms of the reproducing kernel functions in the space $W_2^m[a,b]$. These expressions can be represented by piecewise polynomial of degree $2m-1$. It is worthwhile to point out that the reproducing kernel function has a unique representation. Further, we will give some corollaries and remarks related to these kernel functions. At the end of this section, several examples of such kernel functions are given in the space $W_2^1[0,1]$.

By re-defining the inner product of a RKHS, the reproducing kernel functions of that space can be represented by form of polynomials without changing any other conditions, the higher order of the derivatives and the simpler of the reproducing kernel function expressions. Such expressions of reproducing kernel functions are the simplest from the computational point of view, resulting in speed and accuracy significant improvement in scientific and engineering applications. The performance of such reproducing kernel functions is shown to be very encouraging by experimental results.

The reproducing kernel space $W_2^m[a,b]$ has the simplest reproducing kernel function $K_x(y)$ represented by polynomials and could be applied in many areas. In particular, we will use these reproducing kernel function expressions in order to obtain the analytical approximate solution for a general form of IDEs.

Now, let's find out the expression form of the reproducing kernel function $K_x(y)$ in the space $W_2^m[a,b]$.

Suppose $K_x(y)$ is the reproducing kernel function of the space $W_2^m[a,b]$, then for each fixed $x\in[a,b]$ and any $u(y)\in W_2^m[a,b]$, $y\in[a,b]$ we have $\langle u(y),K_x(y)\rangle=u(x)$. Based on Equations (1.2) and (1.3), we have

$$\langle u(y),K_x(y)\rangle_{W_2^m[a,b]}=\sum_{i=0}^{m-1}u^{(i)}(a)\,K_x^{(i)}(a)+\int_a^b u^{(m)}(y)\,K_x^{(m)}(y)\,dy. \tag{1.7}$$

Applying the integration by parts for the second scheme of the right-hand of Equation (1.7) ,

we obtain

$$\int_a^b u^{(m)}(y)\, K_x^{(m)}(y)\, dy = \sum_{i=0}^{m-1} (-1)^i\, u^{(m-i-1)}(y)\, K_x^{(m+i)}(y)\, |_{y=a}^{b} + \int_a^b (-1)^m\, u(y)\, K_x^{(2m)}(y)\, dy$$

Let $j = m - i - 1$, the first term of the right-hand side of the above formula can be rewritten as

$$\sum_{i=0}^{m-1} (-1)^i\, u^{(m-i-1)}(y)\, K_x^{(m+i)}(y)\, |_{y=a}^{b} = \sum_{j=0}^{m-1} (-1)^{m-j-1}\, u^{(j)}(y)\, K_x^{(2m-j-1)}(y)\, |_{y=a}^{b}\, .$$

After some simplification, Equation (1.7) became's

$$\begin{aligned}\langle u(y), K_x(y)\rangle_{W_2^m[a,b]} &= \sum_{i=0}^{m-1} u^{(i)}(a) \left(K_x^{(i)}(a) - (-1)^{m-i-1} K_x^{(2m-i-1)}(a)\right) + \\ &\quad \sum_{i=0}^{m-1} (-1)^{m-i-1}\, u^{(i)}(b)\, K_x^{(2m-i-1)}(b) + \int_a^b (-1)^m\, u(y)\, K_x^{(2m)}(y)\, dy\end{aligned}$$

Since $K_x(y), u(y) \in W_2^m[a,b]$, it follows that

$$K_x^{(i)}(a) - (-1)^{m-i-1} K_x^{(2m-i-1)}(a) = 0, K_x^{(2m-i-1)}(b) = 0, i = 0, 1, ..., m-1.$$

Then $\langle u(y), K_x(y)\rangle_{W_2^m[a,b]} = \int_a^b u(y) \left((-1)^m K_x^{(2m)}(y)\right) dy.$

Now, for each $x \in [a,b]$, if $K_x(y)$ satisfies $(-1)^m K_x^{(2m)}(y) = \delta(x-y)$, where δ is dirac-delta function, then $\langle u(y), K_x(y)\rangle_{W_2^m[a,b]} = \int_a^b u(y)\, \delta(x-y)\, dy = u(x)$. Obviously, $K_x(y)$ is the reproducing kernel of the space $W_2^m[a,b]$.

Therefore, $K_x(y)$ is the solution of the following generalized differential equations:

$$\begin{cases} (-1)^m K_x^{(2m)}(y) = \delta(x-y) \\ K_x^{(i)}(a) - (-1)^{m-i-1} K_x^{(2m-i-1)}(a) = 0, i = 0, 1, ..., m-1 \\ K_x^{(2m-i-1)}(b) = 0, i = 0, 1, ..., m-1. \end{cases} \tag{1.8}$$

While $x \neq y$

$$(-1)^m K_x^{(2m)}(y) = 0. \tag{1.9}$$

with the boundary conditions (BC's):

$$K_x^{(i)}(a) - (-1)^{m-i-1} K_x^{(2m-i-1)}(a) = 0,\ K_x^{(2m-i-1)}(b) = 0,\ i = 0, 1, ..., m-1 \tag{1.10}$$

The characteristic equation of Equation (1.9) is $\lambda^{2m} = 0$, and their characteristic values are $\lambda = 0$ with $2m$ multiple roots. So, the general solution of Equation (1.9) is as follows:

$$K_x(y) = \begin{cases} \sum_{i=0}^{2m-1} p_i(x) y^i, & y \leq x; \\ \sum_{i=0}^{2m-1} q_i(x) y^i, & y > x. \end{cases} \tag{1.11}$$

On the other hand, since $(-1)^m K_x^{(2m)}(y) = \delta(x-y)$, we have

$$K_x^{(i)}(x+0) = K_x^{(i)}(x-0),\ i = 0, 1, ..., 2m-2. \tag{1.12}$$

Integrating $(-1)^m K_x^{(2m)}(y) = \delta(x-y)$ from $x - \varepsilon$ to $x + \varepsilon$ with respect to y and let $\varepsilon \to 0$, we have the jump degree of $K_x^{(2m-1)}(y)$ at $y = x$ given by

$$(-1)^m \left(K_x^{(2m-1)}(x+0) - K_x^{(2m-1)}(x-0)\right) = 1. \tag{1.13}$$

Equations (1.12) and (1.13) provided $2m$ conditions for solving the coefficients $p_i(x)$ and $q_i(x)$, $i = 0, 1, ..., 2m-1$, in Equation (1.11). Further, Equation (1.10) provided $2m$ BC's. So, we have $4m$ equations. It is easy to know that these $4m$ equations are linear equations with the variables $p_i(x)$ and $q_i(x)$, and the unknown coefficients $p_i(x)$ and $q_i(x)$ of Equation (1.11) could be calculated out by many methods such as Green's function method or by using Mathematica 7.0 software package.

As long as the coefficients $p_i(x)$ and $q_i(x)$ are known, the exact expression of the

producing kernel function $K_x(y)$ of the space $W_2^m[a,b]$ could be calculated out from Equation (1.11). The expression of $K_x(y)$ is a piecewise polynomial with $2m-1$ degrees.

The following corollary summarized some important properties of the reproducing kernel $K_x(y)$ and it is easily obtained.

Corollary 1.4.1 *The reproducing kernel $K_x(y)$ is symmetric, unique and $K_x(x) \geq 0$, for any fixed $x \in [a,b]$.*

Proof. *By the reproducing property, we have*

$$K_x(y) = \langle K_x(\cdot), K_y(\cdot)\rangle = \langle K_y(\cdot), K_x(\cdot)\rangle = K_y(x).$$

Now, let $K_x(y)$ and $R_x(y)$ be all the reproducing kernels of the space $W_2^m[a,b]$, then $K_x(y) = \langle K_x(\cdot), R_y(\cdot)\rangle = \langle R_y(\cdot), K_x(\cdot)\rangle = R_y(x)$. By the symmetry of $R_x(y)$, we have the unique representation of $K_x(y)$. For the last condition, we note that $K_x(x) = \langle K_x(\cdot), K_x(\cdot)\rangle = \|K_x(\cdot)\|^2 \geq 0$.

Before we start in the representation of expressions of such reproducing kernel functions, we need the following remarks:

(i) If the functions in space $W_2^m[a,b]$ require more special BC's, *e.g.*, the BC's of the second-order differential equations as follows: $u(a) = \alpha$, $u(b) = \beta$, or $u'(a) = \alpha$, $u'(b) = \beta$; or the linear BC's: $a_1 u(a) + b_1 u'(a) = \alpha$, $a_2 u(b) + b_2 u'(b) = \beta$; or the periodic linear BC's: $u(a) = u(b)$, $u'(a) = u'(b)$. These different kinds of BC's could be contained in space $W_2^m[a,b]$ after homogenization, *i.e.*, we can find the reproducing kernel function $K_x(y)$ which satisfies these BC's.

(ii) If we are re-defining the inner product in Equation (1.2) of the space $W_2^m[a,b]$ then the unique reproducing kernel function of that space can be represented (not necessary polynomial) by using the same procedure above.

We are ready to present some expressions of reproducing kernel function in the space $W_2^m[0,1]$ with respect to different norm by using the approaches proposed in this section. Thus, from Equation (1.1), we obtain

$$W_2^m[0,1] = \{u(x) : u^{(i)}(x) \text{ is Abs.C}, \ i = 1,2,..,m-1, \ u^{(m)}(x) \in L^2[0,1]\}$$

(A) The space $W_2^1[0,1] = \{u : [0,1] \to \mathbb{R} : u(x)$ is Abs.C and $u'(x) \in L^2[0,1]\}$. The inner product and the norm in the space $W_2^1[0,1]$ are given, respectively, by $\langle u,v\rangle_{W_2^1} = u(0)v(0) + \int_0^1 u'(y)v'(y)\,dy$ and $\|u\|_{W_2^1} = \sqrt{\langle u,u\rangle}$, where $u(x), v(x) \in W_2^1[0,1]$.

To find the reproducing kernel function $K_x(y)$, we apply integration by parts to see that $\langle u,K_x\rangle_{W_2^1} = u(0)K_x(0) + u(y)K_x'(y)|_{y=0}^1 - \int_0^1 u(y)K_x''(y)\,dy.$

Since $u(y), K_x(y) \in W_2^1[0,1]$, we have $K_x(0) - K_x'(0) = 0$ and $K_x'(1) = 0$. Thus, we need to solve the BVP $-K_x''(y) = \delta(x-y)$ subject to $K_x(0) - K_x'(0) = 0$ and $K_x'(1) = 0$. The characteristic equation is $\lambda^2 = 0$, and the characteristic value is $\lambda = 0$ with 2 multiple roots. So, let

$$K_x(y) = \begin{cases} c_1(x) + c_2(x)y, & y \le x, \\ d_1(x) + d_2(x)y, & y > x. \end{cases}$$

From $K_x(0) - K_x'(0) = 0$, $K_x'(1) = 0$, $K_x(x+0) = K_x(x-0)$ and $K_x'(x+0) - K_x'(x-0) = -1$, the unknown coefficients $c_i(x)$ and $d_i(x)$, $i = 1,2$, can be obtained.

In (Yao, 2008), it had been proved that the space $W_2^1[0,1]$ is a complete reproducing kernel space and its reproducing kernel function is

$$K_x(y) = \begin{cases} 1+y, & y \le x, \\ 1+x, & y > x, \end{cases}$$

where $x, y \in [0,1]$. The 3-D image of the reproducing kernel function $K_x(y)$ is shown in Figure 1.1.

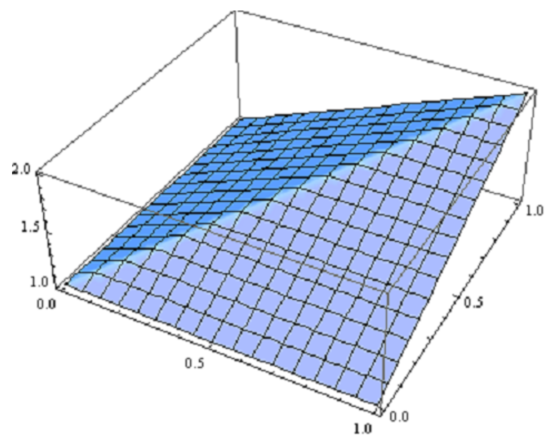


Fig. 1.1. 3-D image of the reproducing kernel $K_x(y)$

(B) The space $W_2^1[0,1] = \{u : u(x)$ is Abs.C, $u'(x) \in L^2[0,1]$ and $u(0) = u(1) = 0\}$. The inner product and the norm in the space $W_2^1[0,1]$ are given, respectively, by $\langle u, v\rangle_{W_2^1} = \int_0^1 u'(y)\, v'(y)\, dy$ and $\|u\|_{W_2^1} = \sqrt{\langle u, u\rangle}$, where $u(x), v(x) \in W_2^1[0,1]$.

Through the integration by parts, we have

$$\langle u, R_x\rangle_{W_2^1} = u(y)\, R_x'(y)\big|_{y=0}^{1} - \int_0^1 u(y)\, R_x''(y)\, dx.$$

However, the space $W_2^1[0,1]$ contains the B.C's $u(0) = u(1) = 0$, then

$$\langle u, R_x\rangle_{W_2^1} = \int_0^1 u(y)\left(-R_x''(y)\right) dy.$$

Now, since $R_x(y) \in W_2^1[0,1]$, then $R_x(0) = R_x(1) = 0$. Thus, we need to solve the BVP $-R_x''(y) = \delta(x-y)$ subject to $R_x(0) = R_x(1) = 0$. The characteristic equation is $\lambda^2 = 0$, and the characteristic value is $\lambda = 0$ with 2 multiple roots.

So, let

$$R_x(y) = \begin{cases} p_1(x) + p_2(x)y, & y \le x, \\ q_1(x) + q_2(x)y, & y > x. \end{cases}$$

From $R_x(0) = R_x(1) = 0$, $R_x(x+0) = R_x(x-0)$ and $R'_x(x+0) - R'_x(x-0) = -1$ we can find the unknown coefficients.

In (Paulsen, 2009), it had been proved that the space $W_2^1[0,1]$ is a complete reproducing kernel space and its reproducing kernel function is

$$R_x(y) = \begin{cases} (1-x)y, & y \le x, \\ (1-y)x, & y > x, \end{cases}$$

where $x, y \in [0,1]$. The 3-D image of the reproducing kernel function $R_x(y)$ is shown in Figure 1.2.

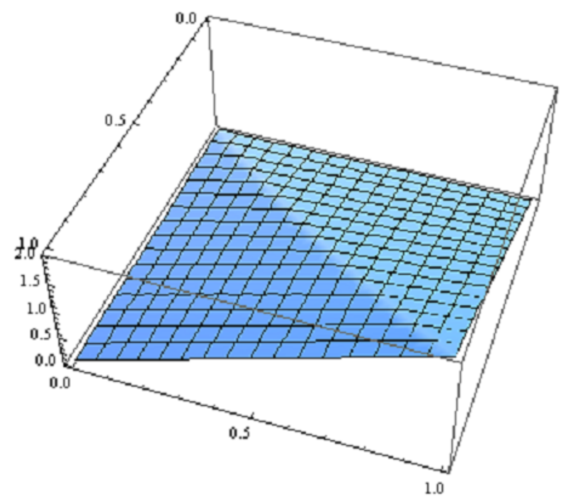


Fig. 1.2. 3-D image of the reproducing kernel $R_x(y)$

(C) Let $W_2^1[0,1]$ be the same set of functions as in Part A. Define a new inner product in the space $W_2^1[0,1]$ by $\langle u,v\rangle_{W_2^1} = \int_0^1 (u(y)v(y) + u'(y)v'(y))\,dy$, where $u(x), v(x) \in W_2^1[0,1]$, and considering the same norm.

Through integration by parts of $\langle u, T_x \rangle_{W_2^1} = \int_0^1 (u(y) T_x(y) + u'(y) T_x'(y))\, dy$,

we have $\langle u, T_x \rangle_{W_2^1} = u(y) T_x'(y)|_{y=0}^{1} + \int_0^1 u(y)(T_x(y) - T_x''(y))\, dy$.

Since there is no B.C's in this space and since $u(y), T_x(y) \in W_2^1[0,1]$, we have $T_x'(0) = T_x'(1) = 0$. Hence, $T_x(y) - T_x''(y) = \delta(x - y)$. The characteristic equation is $1 - \lambda^2 = 0$, and the characteristic values are $\lambda = -1$, $\lambda = 1$. So, let

$$T_x(y) = \begin{cases} c_1(x)e^{-y} + c_2(x)e^{y}, & y \leq x, \\ d_1(x)e^{-y} + d_2(x)e^{y}, & y > x. \end{cases}$$

From $T_x'(0) = T_x'(1) = 0$, $T_x(x+0) = T_x(x-0)$ and the jump degree of $T_x'(y)$, the unknown coefficients can be obtained.

Cui and Li (2003) proved that $W_2^1[0,1]$ is a reproducing kernel space and its reproducing kernel function is

$$T_x(y) = \frac{1}{2\sinh(1)}\left[\cosh(x + y - 1) + \cosh(|x - y| - 1)\right].$$

The 3-D image of the reproducing kernel function $T_x(y)$ is shown in Figure 1.3.

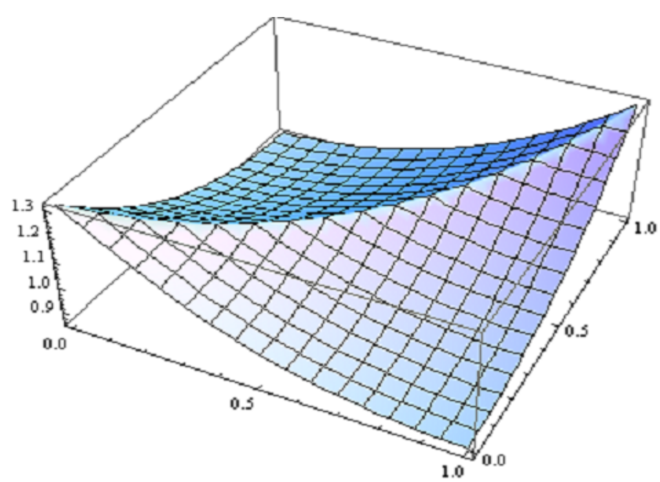


Fig. 1.3. 3-D image of the reproducing kernel $T_x(y)$

1.5 Description of Reproducing Kernel Method

In this section, we will establish an iterative method to construct and calculate the solution for the general m^{th}-order BVP. The formula for calculation and the representation of the exact solution is given in the RKHS. Its approximate solution is obtained by truncating the n-term of the exact solution.

To convey an idea of the RKHS method, we consider the general m^{th}-order BVP of the following type

$$u^{(m)}(x) + a_1(x)\,u^{(m-1)}(x) + ... + a_{m-1}(x)\,u'(x) = F(x, u(x)), \ \ a \leq x \leq b, \qquad (1.14)$$

subject to the BC's

$$\begin{aligned} u^{(i)}(a) &= \alpha_i, \ \ i = 0, 1, 2, ..., r-1, \\ u^{(i)}(b) &= \beta_i, \ \ i = r, r+1, ..., m-1. \end{aligned} \qquad (1.15)$$

where $a_i(x)$, $i = 1, 2, ..., m-1$, are continuous real-valued functions, α_i, $0 \leq i \leq r-1$ and β_i, $r \leq i \leq m-1$ are real finite constants, $u(x)$ is unknown function to be determined, $u^{(m)}(x)$ indicates the m^{th} derivative of $u(x)$, and $F(x, u(x))$ is a linear or nonlinear function depending on the problem discussed.

In order to solve the BVP (1.14) and (1.15) using the RKHS method; First of all, we construct a reproducing kernel space $W_2^{m+1}[a, b]$ in which every function satisfies the homogeneous BC's of Equation (1.15) and then utilize the space $W_2^1[a, b]$. The inner product and the norm in the space $W_2^{m+1}[a, b]$ can be obtained as in Equations (1.2) and (1.3), respectively.

Let $K_x(y)$ and $R_x(y)$ be the reproducing kernel functions of the spaces $W_2^{m+1}[a, b]$ and $W_2^1[a, b]$, respectively, where $K_x(y)$ is a piecewise polynomial with $2m+1$ degrees. Define a differential operator $L : W_2^{m+1}[a, b] \longrightarrow W_2^1[a, b]$ such that $Lu(x) = u^{(m)}(x) + a_1(x)\,u^{(m-1)}(x) + ... + a_{m-1}(x)\,u'(x)$, it is clear that L is bounded linear operator.

Thus, after homogenization of the BC's (1.15), the BVP (1.14) and (1.15) can be converted into the equivalent form as follows:

$$Lu = F(x, u(x)), \ a \leq x \leq b; \tag{1.16}$$

$$u^{(i)}(a) = 0, \ i = 0, 1, 2, ..., r-1; \ u^{(i)}(b) = 0, \ i = r, r+1, ..., m-1. \tag{1.17}$$

where $u(x) \in W_2^{m+1}[a,b]$ and $F(x,u) \in W_2^1[a,b]$ as $u = u(x) \in W_2^{m+1}[a,b]$.

Now, we construct an orthogonal function system of the space $W_2^{m+1}[a,b]$. For a countable dense set $\{x_i\}_{i=1}^{\infty}$ of $[a,b]$, let $\varphi_i(x) = R_{x_i}(x)$. So, from the properties of $R_x(y)$, for every $u(x) \in W_2^1[a,b]$, it follows that $\langle u(x), \varphi_i(x)\rangle_{W_2^1} = \langle u(x), R_{x_i}(x)\rangle_{W_2^1}$ $= u(x_i)$. Additionally, let $\psi_i(x) = L^*\varphi_i(x)$, where L^* is the adjoint operator of L. Obviously, $\psi_i(x) \in W_2^{m+1}[a,b]$. In terms of the properties of $K_x(y)$, one obtains

$$\langle u(x), \psi_i(x)\rangle_{W_2^{m+1}} = \langle u(x), L^*\varphi_i(x)\rangle_{W_2^{m+1}} = \langle Lu(x), \varphi_i(x)\rangle_{W_2^1} = Lu(x_i),$$

where $i = 1, 2, ...$

The following lemmas are collected for future use.

Lemma 1.5.1 *$\psi_i(x)$ can be expressed in the form $\psi_i(x) = L_y K_x(y)|_{y=x_i}$. The subscript y by the operator L indicates that the operator L applies to the function of y.*

Proof. *From the above assumption, it is clear that*

$$\begin{aligned} \psi_i(x) &= L^*\varphi_i(x) = \langle L^*\varphi_i(x), \ K_x(y)\rangle_{W_2^{m+1}} \\ &= \langle \varphi_i(x), LK_x(y)\rangle_{W_2^1} = L_y K_x(y)|_{y=x_i}. \end{aligned}$$

Lemma 1.5.2 $\psi_i(a) = \psi_i(b) = 0, \ i = 1, 2, ...$

Proof. $\psi_i(a) = \langle \psi_i(y), K_a(y)\rangle_{W_2^{m+1}} = \langle L^*\varphi_i(y), K_a(y)\rangle_{W_2^{m+1}} = \langle \varphi_i(y), L_y K_a(y)\rangle_{W_2^1}$.

By the symmetry of $K_a(y)$, we arrive at $K_a(y) = K_y(a) = 0$, thus $\psi_i(a) = 0$.

Similarly, we further obtain $\psi_i(b) = 0$.

Theorem 1.5.1 *Suppose that the inverse operator* L^{-1} *in Equation* (1.16) *exists. Thus, if* $\{x_i\}_{i=1}^{\infty}$ *is dense on* $[a,b]$, *then* $\{\psi_i(x)\}_{i=1}^{\infty}$ *is the complete function system of the space* $W_2^{m+1}[a,b]$.

Proof. *For each fixed* $u(x) \in W_2^{m+1}[a,b]$, *let* $\langle u(x), \psi_i(x)\rangle = 0$, $i = 1, 2, \ldots$, *that is*

$$\langle u(x), \psi_i(x)\rangle_{W_2^{m+1}} = \langle u(x), L^*\varphi_i(x)\rangle_{W_2^{m+1}} = \langle Lu(x), \varphi_i(x)\rangle_{W_2^1} = Lu(x_i) = 0.$$

Note that $\{x_i\}_{i=1}^{\infty}$ *is dense on* $[a,b]$, *therefore,* $Lu(x) = 0$. *It follows that* $u(x) = 0$ *from the existence of* L^{-1} *and the continuity of* $u(x)$. *The proof is complete.*

The orthonormal function system $\{\overline{\psi}_i(x)\}_{i=1}^{\infty}$ of the space $W_2^{m+1}[a,b]$ can be derived from Gram-Schmidt orthogonalization process of $\{\psi_i(x)\}_{i=1}^{\infty}$ as follows:

$$\overline{\psi}_i(x) = \sum_{k=1}^{i} \beta_{ik}\psi_k(x), \ \ i = 1, 2, \ldots \tag{1.18}$$

where β_{ik} are orthogonalization coefficients and are given by

$$\beta_{11} = \frac{1}{\|\psi_1\|}, \quad \beta_{ii} = \frac{1}{\sqrt{\|\psi_i\|^2 - \sum_{k=1}^{i-1} c_{ik}^2}}, \quad \beta_{ij} = \frac{-\sum_{k=j}^{i-1} c_{ik}\beta_{kj}}{\sqrt{\|\psi_i\|^2 - \sum_{k=1}^{i-1} c_{ik}^2}}, \ j < i, \tag{1.19}$$

in which $c_{ik} = \langle \psi_i, \overline{\psi}_k\rangle_{W_2^{m+1}}$ and $\{\overline{\psi}_i(x)\}_{i=1}^{\infty}$ is the orthonormal system in the space $W_2^{m+1}[a,b]$.

Now, we will give the representation of the exact solution of the BVP (1.14) and (1.15) in the space $W_2^{m+1}[a,b]$. Afterward, the convergence of approximate solution $u_n(x)$ to the analytical solution will be obtained.

Theorem 1.5.2 *For each* $u(x) \in W_2^{m+1}[a,b]$, *the series* $\sum_{i=1}^{\infty} \langle u(x), \overline{\psi}_i(x)\rangle \overline{\psi}_i(x)$ *is convergent in sense of the norm of* $W_2^{m+1}[a,b]$. *On the other hand, if* $\{x_i\}_{i=1}^{\infty}$ *is dense*

on $[a,b]$ *and the solution of the BVP* (1.14) *and* (1.15) *is unique, then this solution satisfies the form:*

$$u(x)=\sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}F(x_k,u(x_k))\overline{\psi}_i(x). \tag{1.20}$$

Proof. *Applying Theorem* 1.5.1, *it is easy to see that* $\{\overline{\psi}_i(x)\}_{i=1}^{\infty}$ *is the complete orthonormal basis of the space* $W_2^{m+1}[a,b]$. *Thus,* $u(x)$ *can be expanded in the Fourier series about the orthonormal system* $\{\overline{\psi}_i(x)\}_{i=1}^{\infty}$ *as* $u(x)=\sum_{i=1}^{\infty}\langle u(x),\overline{\psi}_i(x)\rangle\overline{\psi}_i(x)$. *Moreover, the space* $W_2^{m+1}[a,b]$ *is Hilbert space, then the series* $\sum_{i=1}^{\infty}\langle u(x),\overline{\psi}_i(x)\rangle\overline{\psi}_i(x)$ *is convergent in sense of the norm of* $W_2^{m+1}[a,b]$.

Since $\langle v(x),\varphi_i(x)\rangle=v(x_i)$ *for each* $v(x)\in W_2^1[a,b]$. *Hence, we have*

$$\begin{aligned}
u(x) &= \sum_{i=1}^{\infty}\langle u(x),\overline{\psi}_i(x)\rangle_{W_2^{m+1}}\overline{\psi}_i(x)\\
&= \sum_{i=1}^{\infty}\langle\, u(x),\sum_{k=1}^{i}\beta_{ik}\psi_k(x)\,\rangle_{W_2^{m+1}}\overline{\psi}_i(x)\\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\langle u(x),\psi_k(x)\rangle_{W_2^{m+1}}\overline{\psi}_i(x)\\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\langle u(x),L^*\varphi_k(x)\rangle_{W_2^{m+1}}\overline{\psi}_i(x)\\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\langle Lu(x),\varphi_k(x)\rangle_{W_2^{1}}\overline{\psi}_i(x)\\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\langle F(x,u(x)),\varphi_k(x)\rangle_{W_2^{1}}\overline{\psi}_i(x)\\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}F(x_k,u(x_k))\overline{\psi}_i(x).
\end{aligned}$$

Remark 1.5.1 *If Equation* (1.14) *is linear, that is,* $F(x,u(x))=f(x)$. *Then, the analytical and the approximate solution to Equation* (1.14) *can be obtained directly from Equation* (1.20).

We denote the n-term approximate solution to $u(x)$ by

$$u_n(x)=\sum_{i=1}^{n}\sum_{k=1}^{i}\beta_{ik}F(x_k,u(x_k))\overline{\psi}_i(x), \tag{1.21}$$

where $\overline{\psi}_i(x)$ and β_{ik} are given in Equations (1.18) and (1.19), respectively.

Remark 1.5.2 *If Equation* (1.14) *is nonlinear. Then, the approximation solution to Equation* (1.14) *can be obtained using the following iteration method.*

According to Equation (1.20), the representation of the solution of Equation (1.14) can be denoted by

$$u(x) = \sum_{i=1}^{\infty} A_i \overline{\psi}_i(x), \tag{1.22}$$

where $A_i = \sum_{k=1}^{i} \beta_{ik} F(x_k, u(x_k))$.

In fact, A_i, $i = 1, 2, ..$, are unknown, we will approximate A_i using known B_i. Let $u(x_1) = 0$, it follows that $F(x_1, u(x_1))$ is known. So, for a numerical computation, we put $u_0(x_1) = u(x_1)$ and define the n-term approximation to $u(x)$ by

$$u_n(x) = \sum_{i=1}^{n} B_i \overline{\psi}_i(x), \tag{1.22}$$

where the coefficients B_i, $i = 1, 2, .., n$, are given as

$$\begin{cases} B_1 = \beta_{11} F(x_1, u_0(x_1)), \\ u_1(x) = B_1 \overline{\psi}_1(x); \\ B_2 = \sum_{k=1}^{2} \beta_{2k} F(x_k, u_{k-1}(x_k)), \\ u_2(x) = B_1 \overline{\psi}_1(x) + B_2 \overline{\psi}_2(x); \\ \cdots \\ B_N = \sum_{k=1}^{n} \beta_{nk} F(x_k, u_{k-1}(x_k)). \end{cases} \tag{1.23}$$

Remark 1.5.3 *In the iteration process of Equation* (1.22), *we can guarantee that the approximation* $u_n(x)$ *satisfies the BC's of Equation* (1.14).

Indeed, the solution of Equation (1.14) is considered as the fixed point of the following functional under the suitable choice of the initial term $u_0(x)$:

$$\begin{aligned} u_{n+1}(x) &= L^{-1}F(x, u_n(x)) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}F(x_k, u_n(x_k))\overline{\psi}_i(x). \end{aligned}$$

As a well known powerful tool, we have the Banach's fixed point theorem

Theorem 1.5.3 *Assume that X is a Banach space and $A : X \longrightarrow X$ is a nonlinear mapping, and suppose that*

$$\|A[u] - A[v]\| \leq \alpha\|u - v\|, \quad u, v \in X$$

for some constants $\alpha < 1$. Then A has a unique fixed point. Furthermore, the sequence $u_{n+1}(x) = A[u_n]$, with an arbitrary choice of $u_0 \in X$, converges to the fixed point of A.

According to above Theorem, for the nonlinear mapping

$$\begin{aligned} A[u(x)] &= L^{-1}F(x, u(x)) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}F(x_k, u(x_k))\overline{\psi}_i(x), \end{aligned}$$

a sufficient condition for convergence of the present iteration method is strictly contraction of A. Furthermore, the sequence (1.22) converges to the fixed point of A which is also the solution of Equation (1.14).

However, the approximate solution $u_n^N(x)$ can be obtained by taking finitely many terms in the series representation of $u_n(x)$ and

$$u_n^N(x) = \sum_{i=1}^{N}\sum_{k=1}^{i}\beta_{ik}F(x_k, u_{n-1}(x_k))\overline{\psi}_i(x).$$

1.6 Software Libraries

The theory of RKHS started with the two papers of 1921 and 1922 which dealt with the typical reproducing kernels of the Szegö kernel and the Bergman kernel, respectively, and then the theory developed as a large and deep theory in complex analysis by many mathematicians. Meanwhile, the general theory of RKHS was established in a complete form by N. Aronszajn in 1950. Furthermore, the general theory was developed remarkably by L. Schwartz in 1964.

In (Saitoh, 1983), he showed clearly its importance and fundamental concept by combining the general theory and linear mappings in the framework of Hilbert spaces; then the general theory was not restricted to the Bergman and Szegö kernels, but the general theory is important as in the concept of Hilbert spaces.

Software packages have great capabilities for solving IDEs. Sometimes, it is difficult to solve them analytically so it is required to obtain an efficient approximate solution. Thus, some software mathematical packages such as Mathematica or MathCad can be helpful in visualizing the behavior of the solutions of IDEs. Indeed, throughout the whole thesis we used Mathematica 7.0 software package for numerical experiment.

In this section, based on RKHS method, we will introduce an effective algorithm for the following nonlinear IDE

$$\begin{aligned} u'(x) &= F(x,u(x)), 0 \leq x \leq 1; \\ u(0) &= 0. \end{aligned} \tag{1.24}$$

where $F(x,u(x)) = f(x) + \int_0^x h(x,t)N(u(t))dt \in W_2^1[0,1]$, $f(x)$ and $h(x,t)$ are known functions, $N(u(x))$ is a nonlinear function of u, $u(x) \in W_2^2[0,1]$ is unknown function to be determined. We suppose that IDE (1.24) has a unique solution.

The space $W_2^2[0,1]$ is a Hilbert Space with some special properties. So, all the properties of the Hilbert Space will be hold. Further, the space $W_2^2[0,1]$ possesses some special and better properties which could make some problems be solved easier. For

instance, many problems studied in $L_2[0,1]$, which is a complete Hilbert space, requires large amount of integral computations, and such computations may be very difficult in some cases. Thus, the numerical integrals have to be calculated in the cost of losing some accuracy. However, the properties of the space $W_2^2[0,1]$ require no more integral computation for some functions, instead of computing some values of a function at some nodes. This simplification of integral computation not only improves the computational speed, but also improves the computational accuracy.

The aim of the next algorithm is to implement a procedure to solve IDE (1.24) based on RKHS method that described in Section 1.5.

Algorithm 1.1: To approximate the solution of the IDE (1.24) based on RKHS method, there are five main steps:

Input: Integer n; The functions $f(x)$, $k_1(x,y)$ and $k_2(x,y)$; The differential operator L; The inner product $\langle u(x), v(x)\rangle_{W_2^2}$.

Output: Approximate solution $u_n(x)$ of the IDE (1.24).

The steps of the algorithm are

Step A: Fixed x and set $x, y \in [0,1]$;

For $i = 1, 2, ..., n$ do steps (1, 2 & 3);

Step 1: Set $x_i = \frac{i-1}{n-1}$;

Step 2: If $y \leq x$ then set $K(x,y) = k_1(x,y)$

else set $K(x,y) = k_2(x,y)$;

Step 3: Set $\psi_i(x) = L_y[K(x,y)]\mid_{y=x_i}$;

Output the orthogonal function system $\psi_i(x)$.

Step B: For $i = 1, 2, ..., n$;

For $j = 1, 2, ..., i$ set $c_{ij} = \left\langle \psi_j(x), \psi_i(x) \right\rangle_{W_2^2}$;

Set $\beta_{11} = 1/Sqrt(c_{11})$;

Output c_{ij} and β_{11}.

Step C: For $i = 2, 3..., n$ do steps (1° & 2°);

Step 1°: For $k = 1, 2..., i-1$ set $cc_{ik} = \sum\limits_{m=1}^{k} \beta_{km} c_{im}$;

Step 2°: For $j = 1, 2..., i$. If $j \neq i$

then set $\beta_{ij} = -\sum\limits_{k=j}^{i-1} cc_{ik}\beta_{kj} / \sqrt{c_{ii} - \sum\limits_{k=1}^{i-1} cc_{ik}^2}$

else set $\beta_{ii} = 1/\sqrt{c_{ii} - \sum\limits_{k=1}^{i-1} cc_{ik}^2}$;

Output the orthogonalization coefficients β_{ij}.

Step D: For $i = 1, 2, ..., n$ set $\overline{\psi_i}(x) = \sum\limits_{k=1}^{i} \beta_{ik}\psi_i(x)$;

Output the orthonormal function system $\overline{\psi_i}(x)$.

Step E: Set $u_0(x_1) = 0$;

For $i = 1, 2, ...n$ do steps (1^*, 2^* & 3^*);

Step 1*: Set $u(x_i) = u_{i-1}(x_i)$;

Step 2*: Set $B_i = \sum\limits_{k=1}^{i} \beta_{ik} F(x_k, u_{k-1}(x_k))$

Step 3*: Set $u_i(x) = \sum\limits_{k=1}^{i} B_k \overline{\psi_k}(x)$.

The n-term approximate solution $u_n(x)$ of IDE (1.24) is obtained.

Chapter Two

First-Order Integro-Differential Equation

Mathematical modeling of real-life, physics and engineering problems usually results in functional equations. For example, PDEs, integral and IDEs, stochastic equations and others. Many mathematical formulation of physical phenomena contain IDEs, these equations arise in fluid dynamics, biological models and chemical kinetics; for more details see (Kythe & Puri, 2002) and (Wazwaz, 2006) and the references cited therein. IDEs are usually difficult to solve analytically so it is required to obtain an efficient approximate solution. Therefore, they have been of great interest by several authors.

Yang and Cui (2006) obtained the solution of a class of IDEs by using the RKHS, they transformed the IDEs into the linear equations. Recently, the authors (Yulan, et al., 2009) presented a new algorithm for giving the analytical and approximate solutions of Fredholm-Volterra IDE in the RKHS.

In this chapter, the RKHS method is applied to approximate the solution of a general form of first-order IDE. It is a relatively new analytical technique. The analytical and approximate solutions are represented in terms of series in the RKHS. The n-term approximation is obtained and is proved to converge to the analytical solution. In the meantime, an iterative method of obtaining the solution is presented in the RKHS. Further, implementations of the method on a nonlinear IDE of Volterra type

$$u'(x) = f(x) + \int_a^x F(x, t, u(t), u'(t))\, dt,\ \ x \in [a, b] \tag{2.1}$$

and Fredholm type

$$u'(x) = f(x) + \int_a^b G(x, t, u(t), u'(t))\, dt,\ \ x \in [a, b] \tag{2.2}$$

will be given in the space $W_2^2[a, b]$.

The proposed method has an advantage that it is possible to pick any point in the interval of integration and as well the approximate solutions and its derivative will be applicable.

2.1 Volterra Integro-Differential Equation

This section investigates the solutions of the first-order Volterra IDEs using the RKHS method. The analytical solution $u(x)$ and approximate solution $u_n(x)$ are represented in the form of series in the space $W_2^2[a,b]$. Afterward, a method for the existence of solutions for Equation (2.1) based on RKHS is described. Moreover, we give an iterative method to solve a nonlinear first-order Volterra IDE. The error of the approximate solution is monotone decreasing in sense of the norm of $W_2^2[a,b]$. Various numerical examples are presented to illustrate the computational efficiency and the accuracy of the proposed method at the end of this section.

2.1.1 Introduction

We consider the Volterra IDE of the following form:

$$\begin{aligned} &u'(x) = F(x, u(x), Tu(x)), \ a \leq x \leq b; \\ &u(a) = u_0, \end{aligned} \tag{2.3}$$

where $Tu(x) = \displaystyle\int_a^x h(x,t)u(t)\,dt$, a, b and u_0 are real finite constants, $h(x,t)$ is known continuous function, $u(x)$ is unknown function to be determined and $F(x,t,u(t),Tu(x))$ is a linear or nonlinear function depending on the problem discussed. We suppose that IDE (2.3) have a unique solution.

Now, we construct several reproducing kernels in order to solve IDE (2.3). The space $W_2^2[a,b]$ is defined as

$$W_2^2[a,b] = \{u : u, u' \text{ is Abs.C}, \ u, u', u'' \in L^2[a,b], \ u(a) = 0\}.$$

The inner product and the norm in the space $W_2^2[a,b]$ are defined respectively by

$$\langle u,v\rangle_{W_2^2}=u(a)\,v(a)+u'(a)\,v'(a)+\int_a^b u''(t)\,v''(t)\,dt, \tag{2.4}$$

and $\|u\|_{W_2^2}=\sqrt{\langle u,u\rangle_{W_2^2}}$ where $u,v\in W_2^2[a,b]$. It is easy to show that the space $W_2^2[a,b]$ is a complete reproducing kernel space. That is, for each fixed $x\in[a,b]$, there exists $K_x(y)\in W_2^2[a,b]$ such that $\langle u(y),K_x(y)\rangle_{W_2^2}=u(x)$ for any $u(y)\in W_2^2[a,b]$ and $y\in[a,b]$. The reproducing kernel $K_x(y)$ can be written as

$$K_x(y)=\begin{cases}\sum\limits_{i=1}^{4}c_iy^{i-1}, & y\le x,\\ \sum\limits_{i=1}^{4}d_iy^{i-1}, & y>x.\end{cases}$$

The method of obtaining the reproducing kernel $K_x(y)$ is given in Section 1.4. Therefore, the unique representation of the reproducing kernel $K_x(y)$ is provided by

$$K_x(y)=\begin{cases}\frac{1}{6}(y-a)\left(2a^2-y^2+3x(2+y)-a(6+3x+y)\right), & y\le x,\\ \frac{1}{6}(x-a)\left(2a^2-x^2+3y(2+x)-a(6+x+3y)\right), & y>x.\end{cases} \tag{2.5}$$

We mention here that the kernel $K_x(y)$ is symmetric, that is $K_x(y)=K_y(x)$ for each x and y, and $K_x(x)\ge 0$ for any fixed $x\in[a,b]$.

Furthermore, if we are re-defining the inner product (2.3) by

$$\langle u,v\rangle_{W_2^2}=u(a)\,v(a)+u(b)\,v(b)+\int_a^b u''(t)\,v''(t)\,dt,\quad u,v\in W_2^2[a,b], \tag{2.6}$$

then the space $W_2^2[a,b]$ is a RKHS and its reproducing kernel is given by

$$R_x(y)=\frac{1}{6(a-b)^2}\begin{cases}\begin{pmatrix}-2a^3(b-x)(b-y)+a^2(6+2b^3+x^3+3xy^2\\ -3b(x^2+y^2))+y(-3b^2x^2+bx^3-b^2y^2+x\\ \left(6+2b^3+by^2\right))-a(\left(-3bx^2+x^3\right)(b+y)+y\\ \left(6+2b^3-3b^2y-by^2\right)+x(6+2b^3+3by^2+y^3))\end{pmatrix}, & y\leq x,\\ \begin{pmatrix}-2a^3(b-x)(b-y)+a^2(6+2b^3+3x^2y+y^3\\ -3b(x^2+y^2))+x(6y+2b^3y+by\left(x^2+y^2\right)-\\ b^2\left(x^2+3y^2\right))+a(\left(3bx^2+x^3\right)(b-y)-y(6+2b^3\\ -3b^2y+by^2)-x\left(6+2b^3-3by^2+y^3\right))\end{pmatrix}, & y>x.\end{cases} \tag{2.7}$$

The space $W_2^1[a,b]$ is defined by $W_2^1[a,b]=\{u : u \text{ is Abs.C}, u,u'\in L^2[a,b]\}$. The inner product and the norm in the space $W_2^1[a,b]$ are given, respectively, by

$$\langle u,v\rangle_{W_2^1}=\int_a^b u(t)\,v(t)+u'(t)\,v'(t)\,dt, \tag{2.8}$$

and $\|u\|_{W_2^1}=\sqrt{\langle u,u\rangle_{W_2^1}}$ where $u,v\in W_2^1[a,b]$. Recall that $W_2^1[a,b]$ is also a complete reproducing kernel space and its reproducing kernel is given by

$$T_x(y)=\frac{1}{2\sinh(b-a)}\left[\cosh(x+y-b-a)+\cosh(|x-y|-b+a)\right].$$

From the definition of the reproducing kernel spaces $W_2^1[a,b]$ and $W_2^2[a,b]$, we get $W_2^1[a,b]\supset W_2^2[a,b]$, for any $u\in W_2^2[a,b]$, and $\|u\|_{W_2^1}\leq\|u\|_{W_2^2}$.

2.1.2 The Analytical Solution and Theoretical Basis

In this section, representation of the solution of Equation (2.3) is given in the reproducing kernel space $W_2^2[a,b]$. We define a differential operator $L: W_2^2[a,b]\longrightarrow W_2^1[a,b]$ such that $Lu(x)=u'(x)$.

After homogenization of the initial condition of Equation (2.3), the IDE (2.3) can

be converted into the equivalent form as follows:

$$\begin{cases} Lu(x) = F(x, u(x), Tu(x)), \\ u(a) = 0; \end{cases} \tag{2.9}$$

where $x \in [a, b]$, $u(x) \in W_2^2[a, b]$ and $F(x, u(x), Tu(x)) \in W_2^1[a, b]$.

Lemma 2.1.1 *L is a bounded operator from $W_2^2[a, b]$ to $W_2^1[a, b]$.*

Proof. *By reproducing property of $K_x(y)$ and Schwarz inequality, we obtain* $|u(x)| = |\langle u(y), K_x(y)\rangle|_{W_2^2} \leq \|K_x(y)\|_{W_2^2} \|u(y)\|_{W_2^2} \leq M_0 \|u(y)\|_{W_2^2}$.

From the representation of $K_x(y)$, we can get $|u'(x)| = \left|\langle u(y), K'_x(y)\rangle_{W_2^2}\right| \leq \|K'_x(y)\|_{W_2^2} \|u(y)\|_{W_2^2}$.

Since $K_x^{(i)}(y)$, $i = 1, 2$ is uniformly bounded about x and y, we have $\left|u^{(i)}(x)\right| \leq M_i \|u(y)\|_{W_2^2}$. *Hence,* $\|Lu(x)\|_{W_2^1}^2 = \|u'(x)\|_{W_2^1}^2 = \int_a^b (u'(t))^2 + (u''(t))^2\, dt \leq M \|u(x)\|_{W_2^2}^2$, *where* $M = (b - a)(M_1^2 + M_2^2)$.

Now, we construct an orthogonal function system of $W_2^2[a, b]$, let $\varphi_i(x) = T_{x_i}(x)$ and $\psi_i(x) = L^*\varphi_i(x)$, where $\{x_i\}_{i=1}^{\infty}$ is dense in the interval $[a, b]$ and L^* is the adjoint operator of L. From the properties of the reproducing kernel $T_x(y)$, we have $\langle u(x), \varphi_i(x)\rangle_{W_2^1} = u(x_i)$. In terms of the properties of $K_x(y)$, one obtains

$$\langle u(x), \psi_i(x)\rangle_{W_2^3} = \langle u(x), L^*\varphi_i(x)\rangle_{W_2^3} = \langle Lu(x), \varphi_i(x)\rangle_{W_2^1} = Lu(x_i). \tag{2.10}$$

Furthermore, $\psi_i(x)$ can be expressed in the form $\psi_i(x) = L_y K_x(y)\,|_{y=xi}$. That is,

$$\begin{aligned} \psi_i(x) &= L^*\varphi_i(x) = \langle L^*\varphi_i(x), K_x(y)\rangle_{W_2^3} = \langle \varphi_i(x), LK_x(y)\rangle_{W_2^3} \\ &= L_y K_x(y)\,|_{y=xi} = F(x_i, K_x(x_i), T[K_x(x_i)]), \end{aligned}$$

where the symbol L_y indicates that the operator L applies to the function of y.

For Equation (2.9), suppose that the inverse operator L^{-1} exists, therefore, if $\{x_i\}_{i=1}^{\infty}$ is dense on $[a,b]$, then $\{\psi_i(x)\}_{i=1}^{\infty}$ is the complete function system of the space $W_2^2[a,b]$. Practise Gram-Schmidt orthonormalization for $\{\psi_i(x)\}_{i=1}^{\infty}$, we get $\overline{\psi}_i(x) = \sum_{k=1}^{i} \beta_{ik}\psi_k(x)$, where β_{ik} are coefficients of Gram-Schmidt orthonormalization and are given by Equation (1.19), and $\{\overline{\psi}_i(x)\}_{i=1}^{\infty}$ is the orthonormal system in the space $W_2^2[a,b]$.

Lemma 2.1.2 *If $u(x) \in W_2^2[a,b]$, then there exists $M > 0$ such that $\left|\left|u^{(i)}(x)\right|\right|_C \leq M\,||u(x)||_{W_2^2}$, $i = 0,1$, where $||u(x)||_C = \max\limits_{a\leq x\leq b} |u(x)|$.*

Proof. *For any $x, y \in [a,b]$, we have $u^{(i)}(x) = \langle\, u(y), K_x^{(i)}(y)\,\rangle_{W_2^2}$, $i = 0,1$. By the expression of $K_x(y)$, it follows that $\|\, K_x^{(i)}(\cdot)\,\|_{W_2^2} \leq M_i, i = 0,1$. Thus*

$$\left|u^{(i)}(x)\right| = \left|\langle\, u(x), K_x^{(i)}(x)\,\rangle_{W_2^2}\right| \leq \|u(x)\|_{W_2^2}\, \|\, K_x^{(i)}(x)\,\|_{W_2^2} \leq M_i\,\|u(x)\|_{W_2^2}.$$

Hence, $\left|\left|u^{(i)}(x)\right|\right|_C \leq M\ ||u(x)||_{W_2^2}\,(i = 0,1)$, where $M = \max\{M_0, M_1\}$.

Theorem 2.1.1 *If $\{x_i\}_{i=1}^{\infty}$ is dense on $[a,b]$, then for IDE (2.3)*

(i) *the exact solution of Equation (2.9) could be represented by*

$$u(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} F(x_k, u(x_k), Tu(x_k))\,\overline{\psi}_i(x), \tag{2.11}$$

(ii) *the approximate solution*

$$u_n(x) = \sum_{i=1}^{n}\sum_{k=1}^{i} \beta_{ik} F(x_k, u(x_k), Tu(x_k))\,\overline{\psi}_i(x), \tag{2.12}$$

and $u_n'(x)$ are converging uniformly to the exact solution $u(x)$ and its derivative $u'(x)$ respectively, as $n \to \infty$.

Proof. *Let $u(x)$ be solution of Equation (2.9) in the space $W_2^2[a,b]$.*

(i) *Since* $\{\overline{\psi}_i(x)\}_{i=1}^{\infty}$ *is an orthonormal system,* $u(x)$ *can be expanded in the Fourier series as*

$$u(x)=\sum_{i=1}^{\infty}\left\langle u(x),\overline{\psi}_i(x)\right\rangle_{W_2^2}\overline{\psi}_i(x). \tag{2.13}$$

Since the space $W_2^2[a,b]$ *is Hilbert, then the series* $\sum_{i=1}^{\infty}\left\langle u(x),\overline{\psi}_i(x)\right\rangle\overline{\psi}_i(x)$ *is convergent in sense of the norm of* $W_2^2[a,b]$. *Substituting* $\overline{\psi}_i(x)=\sum_{k=1}^{i}\beta_{ik}\psi_k(x)$ *into Equation* (2.13), *it holds that*

$$\begin{aligned}
u(x) &= \sum_{i=1}^{\infty}\left\langle u(x),\overline{\psi}_i(x)\right\rangle_{W_2^2}\overline{\psi}_i(x)=\sum_{i=1}^{\infty}\langle\, u(x),\sum_{k=1}^{i}\beta_{ik}\psi_k(x)\,\rangle_{W_2^2}\overline{\psi}_i(x)\\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\left\langle u(x),\psi_k(x)\right\rangle_{W_2^2}\overline{\psi}_i(x)=\sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\left\langle u(x),L^*\varphi_k(x)\right\rangle_{W_2^2}\overline{\psi}_i(x)\\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\left\langle Lu(x),\varphi_k(x)\right\rangle_{W_2^1}\overline{\psi}_i(x)=\sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}Lu(x_k)\overline{\psi}_i(x)\\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}F(x_k,u(x_k),Tu(x_k))\overline{\psi}_i(x).
\end{aligned}$$

Therefore, the form of Equation (2.11) *is the exact solution of Equation* (2.9).

(ii) *By Lemma* 2.1.2, *for any* $x\in[a,b]$, *we get*

$$\begin{aligned}
|u_n(x)-u(x)| &= \left|\left\langle u_n(x)-u(x),K_x(x)\right\rangle_{W_2^2}\right|\le\|K_x(x)\|_{W_2^2}\|u_n(x)-u(x)\|_{W_2^2}\\
&\le M_0\|u_n(x)-u(x)\|_{W_2^2}.
\end{aligned}$$

On the other hand

$$\begin{aligned}
|u_n'(x)-u'(x)| &= \left|\left\langle u_n'(x)-u(x),K_x'(x)\right\rangle_{W_2^2}\right|\le\|K_x'(x)\|_{W_2^2}\|u_n'(x)-u(x)\|_{W_2^2}\\
&\le M_1\|u_n(x)-u(x)\|_{W_2^2}.
\end{aligned}$$

Hence, $\left|u_n^{(i)}(x)-u^{(i)}(x)\right|\le\left\|u_n^{(i)}(x)-u^{(i)}(x)\right\|_C\le M_i\left\|u_n^{(i)}(x)-u^{(i)}(x)\right\|_{W_2^2}\longrightarrow 0$, $i=0,1,$ *as* $n\to\infty$. *Thus, the approximate solution* $u_n(x)$ *and* $u_n'(x)$ *converge uniformly to* $u(x)$ *and its derivative* $u'(x)$, *respectively.*

2.1.3 Iterative Method and Convergence Analysis

In this section, an iterative method of obtaining the solution of Equation (2.9) is presented in the RKHS.

Firstly, we mention the following remark in order to solve IDE (2.3) numerically.

If Equation (2.3) is linear, that is, $F(x, u(x), Tu(x)) = f(x)$, then the analytical and approximate solutions can be obtained directly from Equations (2.11) and (2.12), respectively. If Equation (2.3) is nonlinear, then the analytical and approximate solutions can be obtained using the following iterative method.

According to Equation (2.11) the representation of the solution of IDE (2.3) can be denoted by

$$u(x) = \sum_{i=1}^{\infty} A_i \overline{\psi}_i(x), \tag{2.14}$$

where $A_i = \sum_{k=1}^{i} \beta_{ik} F(x_k, u(x_k), Tu(x_k))$.

In fact, A_i, $i = 1, 2, \ldots$, are unknown, we will approximate A_i using the known B_i. For a numerical computation, we define initial function $u_0(x_1) = 0$ and define the n-term approximation to $u(x)$ by

$$u_n(x) = \sum_{i=1}^{n} B_i \overline{\psi}_i(x), \tag{2.15}$$

where the coefficients B_i, $i = 1, 2, \ldots, n$, are given as

$$\begin{cases} B_1 = \beta_{11} F(x_1, u_0(x_1), Tu_0(x_1)), \\ u_1(x) = B_1 \overline{\psi}_1(x), \\ B_2 = \sum_{k=1}^{2} \beta_{2k} F(x_k, u_{k-1}(x_k), Tu_{k-1}(x_k)), \\ u_2(x) = B_1 \overline{\psi}_1(x) + B_2 \overline{\psi}_2(x), \\ \ldots \\ B_n = \sum_{k=1}^{n} \beta_{nk} F(x_k, u_{k-1}(x_k), Tu_{k-1}(x_k)), \end{cases} \tag{2.16}$$

In the iterative process of Equation (2.15), we can guarantee that the approximation $u_n(x)$ satisfies the initial condition of IDE (2.3).

Next, we will prove that $u_n(x)$ in the iterative formula (2.15) is convergent to the exact solution of Equation (2.9). The following lemmas are collected for future use.

Lemma 2.1.3 *If $u_n(x) \to u(x)$ in sense of the norm of $W_2^2[a,b]$ and $x_n \to y$ as $n \to \infty$, then $u_n(x_n) \to u(y)$.*

Proof. *Since*

$$\begin{aligned} |u_n(x_n) - u(y)| &= |u_n(x_n) - u_n(y) + u_n(y) - u(y)| \\ &\leq |u_n(x_n) - u_n(y)| + |u_n(y) - u(y)|. \end{aligned}$$

By reproducing property of $K_x(y)$, we have $u_n(x_n) = \langle u_n(x), K_{x_n}(x)\rangle_{W_2^2}$ and $u_n(y) = \langle u_n(x), K_y(x)\rangle_{W_2^2}$. It follows that $|u_n(x_n) - u_n(y)| = |\langle u_n(x), K_{x_n}(x) - K_y(x)\rangle_{W_2^2}| \leq \|u_n(x)\|_{W_2^2} \|K_{x_n}(x) - K_y(x)\|_{W_2^2}$.

From the symmetry $K_x(y)$, we have $\|K_{x_n}(x) - K_y(x)\|_{W_2^2} \to 0$ as $n \to \infty$. Thus, $|u_n(x_n) - u_n(y)| \to 0$ as soon as $x_n \to y$.

On the other hand, for any $x, y \in [a,b]$, it holds that $|u_n(y) - u(y)|_C \to 0$ as soon as $\|u_n(y) - u(y)\|_{W_2^2} \to 0$ as $n \to \infty$ from Lemma 2 1.2. Therefore, $u_n(x_n) \to u(y)$ as $x_n \to y$ and $n \to \infty$.

By means of the continuation of $Tu(\cdot)$, it is obtained that $Tu_n(x_n) \to Tu_n(y)$ as $x_n \to y$, $n \to \infty$. Hence, by the continuity of F, Lemmas 2.1.2 and 2.1.3, we have

$$F(x_n, u_{n-1}(x_n), Tu_{n-1}(x_n)) \longrightarrow F(y, u(y), Tu(y)) \text{ as } n \to \infty.$$

Lemma 2.1.4 *$\{u_n\}_{n=1}^{\infty}$ in Equation (2.15) is monotone increasing in sense of the norm of $W_2^2[a,b]$.*

Proof. *By Theorem 1.5.1, $\{\overline{\psi}_i\}_{i=1}^{\infty}$ is the complete orthonormal system in the space*

$W_2^2[a,b]$. *Hence, we have*

$$\|u_n\|_{W_2^2}^2 = \langle u_n(x), u_n(x)\rangle_{W_2^2} = \langle \sum_{i=1}^{n} B_i\overline{\psi}_i(x), \sum_{i=1}^{n} B_i\overline{\psi}_i(x)\rangle_{W_2^2} = \sum_{i=1}^{n}(B_i)^2.$$

Therefore, $\|u_n\|_{W_2^2}$ *is monotone increasing.*

Lemma 2.1.5 $Lu_n(x_j) = F(x_j, u_{j-1}(x_j), Tu_{j-1}(x_j))$, $j \le n$.

Proof. *The proof will be obtained by induction as follows. If* $j \le n$, *then*

$$\begin{aligned} Lu_n(x_j) &= \sum_{i=1}^{n} B_i L\overline{\psi}_i(x_j) = \sum_{i=1}^{n} B_i \left\langle L\overline{\psi}_i(x), \phi_j(x)\right\rangle_{W_2^1} \\ &= \sum_{i=1}^{n} B_i \left\langle \overline{\psi}_i(x), L^*\phi_j(x)\right\rangle_{W_2^2} = \sum_{i=1}^{n} B_i \left\langle \overline{\psi}_i(x), \psi_j(x)\right\rangle_{W_2^2}. \end{aligned} \quad (2.17)$$

Multiplying both sides of Equation (2.17) *by* β_{jl}, *summing for* l *from* 1 *to* j *and using the orthogonality of* $\left\{\overline{\psi}_i(x)\right\}_{i=1}^{\infty}$, *yields that*

$$\begin{aligned} \sum_{l=1}^{j} \beta_{jl} Lu_n(x_l) &= \sum_{i=1}^{n} B_i \left\langle \overline{\psi}_i(x), \textstyle\sum_{l=1}^{j}\beta_{jl}\psi_l(x)\right\rangle_{W_2^2} = \sum_{i=1}^{n} B_i \left\langle \overline{\psi}_i(x), \overline{\psi}_j(x)\right\rangle_{W_2^2} \\ &= B_j = \sum_{l=1}^{j} \beta_{jl} F(x_l, u_{l-1}(x_l), Tu_{l-1}(x_l)). \end{aligned}$$

Now, if $j=1$, *then* $Lu_n(x_1) = F(x_1, u_0(x_1), Tu_0(x_1))$. *If* $j=2$, *then*

$$\beta_{21}Lu_n(x_1) + \beta_{22}Lu_n(x_2) = \beta_{21}F(x_1, u_0(x_1), Tu_0(x_1)) + \beta_{22}F(x_2, u_1(x_2), Tu_1(x_2)).$$

It is clear that $Lu_n(x_2) = F(x_2, u_1(x_2), Tu_1(x_2))$. *Furthermore, it is easy to see by induction that* $Lu_n(x_j) = F(x_j, u_{j-1}(x_j), Tu_{j-1}(x_j))$.

Lemma 2.1.6 $Lu_n(x_j) = Lu(x_j)$, $j \le n$.

Proof. *We know* $u_n(x)$ *converge uniformly to* $u(x)$. *It follows that, by taking the limit on both sides of Equation* (2.15); $u(x) = \sum_{i=1}^{\infty} B_i\overline{\psi}_i(x)$. *Therefore* $u_n(x) = P_n u(x)$, *where* P_n *is an orthogonal projector from the space* $W_2^2[a,b]$ *to* Span$\{\psi_1, \psi_2, ..., \psi_n\}$.

Also, $Lu_n(x_j) = \langle Lu_n(x), \phi_j(x)\rangle_{W_2^1} = \langle u_n(x), L^*\phi_j(x)\rangle_{W_2^2} = \langle P_n u(x), \psi_j(x)\rangle_{W_2^2} = \langle u(x), P_n\psi_j(x)\rangle_{W_2^2} = \langle u(x), \psi_j(x)\rangle_{W_2^2} = \langle Lu(x), \phi_j(x)\rangle_{W_2^1} = Lu(x_j)$.

Theorem 2.1.2 *Suppose that* $\|u_n\|_{W_2^2}$ *is bounded in Equation* (2.15) *and* $F(x, u(x), Tu(x)) \in W_2^1[a,b]$. *If* $\{x_i\}_{i=1}^{\infty}$ *is dense in* $[a,b]$, *then the* n-term *approximate solution* $u_n(x)$ *in the iterative formula* (2.15) *converges to the exact solution* $u(x)$ *of Equation* (2.9) *in the space* $W_2^2[a,b]$ *and* $u(x) = \sum_{i=1}^{\infty} B_i\overline{\psi}_i(x)$, *where* B_i *are given by Equation* (2.16).

Proof. *First of all, we will prove the convergence of* $u_n(x)$. *From Equation* (2.15), *we infer that*

$$u_{n+1}(x) = u_n(x) + B_{n+1}\overline{\psi}_{n+1}(x).$$

The orthogonality of $\{\overline{\psi}_i(x)\}_{i=1}^{\infty}$ *yields that*

$$\begin{aligned} ||u_{n+1}||_{W_2^2}^2 &= ||u_n||_{W_2^2}^2 + (B_{n+1})^2 \\ &= ||u_{n-1}||_{W_2^2}^2 + (B_n)^2 + (B_{n+1})^2 = \ldots = ||u_0||_{W_2^2}^2 + \sum_{i=1}^{n+1}(B_i)^2. \end{aligned}$$

From Lemma 2.1.4 *the sequence* $\|u_n\|_{W_2^2}$ *is monotone increasing. Due to the condition that* $\|u_n\|_{W_2^2}$ *is bounded,* $\|u_n\|_{W_2^2}$ *is convergent as* $n \longrightarrow \infty$. *Then, there exists a constant* c *such that* $\sum_{i=1}^{\infty}(B_i)^2 = c$. *It implies that* $B_i = \sum_{k=1}^{i}\beta_{ik}F(x_k, u_{k-1}(x_k), Tu_{k-1}(x_k)) \in l^2$, $i = 1, 2, \ldots$

If $m > n$, *using* $(u_m - u_{m-1}) \perp (u_{m-1} - u_{m-2}) \perp \ldots \perp (u_{n+1} - u_n)$, *then one gets*

$$\begin{aligned} ||u_m(x) - u_n(x)||_{W_2^2}^2 &= ||u_m(x) - u_{m-1}(x) + u_{m-1}(x) - \ldots + u_{n+1}(x) - u_n(x)||_{W_2^2}^2 \\ &= ||u_m(x) - u_{m-1}(x)||_{W_2^2}^2 + \ \ldots \ + \ ||u_{n+1}(x) - u_n(x)||_{W_2^2}^2 \end{aligned}$$

Furthermore, $||u_m(x) - u_{m-1}(x)||_{W_2^2}^2 = (B_m)^2$. *Consequently,* $||u_m(x) - u_n(x)||_{W_2^2}^2 = \sum_{i=n+1}^{m}(B_i)^2 \to 0$ *as* $n, m \to \infty$.

Considering the completeness of $W_2^2[a,b]$, *there exists a* $u(x) \in W_2^2[a,b]$ *such that*

$u_n(x) \longrightarrow u(x)$ *as* $n \to \infty$ *in sense of the norm of* $W_2^2[a,b]$.

Secondly, we will prove that $u(x)$ *is the solutions of IDE* (2.9)*. Since* $\{x_i\}_{i=0}^{\infty}$ *is dense on* $[a,b]$*, for any* $x \in [a,b]$*, there exists subsequence* $\{x_{n_j}\}$ *such that* $x_{n_j} \to x$ *as* $j \to \infty$*. From Lemma* 2.1.5 *and Lemma* 2.1.6*, it is clear that* $Lu(x_{n_j}) = F(x_{n_j}, u_{n_j-1}(x_k), Tu_{n_j-1}(x_k))$*. Hence, let* $j \to \infty$ *in the previous equation, by lemma* 2.1.3 *and the continuity of* F*, we have* $Lu(x) = F(x, u(x), Tu(x))$*. That is,* $u(x)$ *is the solution of IDE* (2.9) *and* $u(x) = \sum\limits_{i=1}^{\infty} B_i \overline{\psi}_i(x)$ *where* B_i *are given by Equation* (2.16)*. The proof is complete.*

Theorem 2.1.3 *Assume* $u(x)$ *is the solution of IDE* (2.3) *and* r_n *is the approximate error of* $u_n(x)$*, where* $u_n(x)$ *is given by* (2.15)*. Then the error* $r_n(x)$ *is monotone decreasing in sense of the norm of* $W_2^2[a,b]$*.*

Proof. *It obvious that* $\|r_n(x)\|_{W_2^2}^2 = \|u(x) - u_n(x)\|_{W_2^2}^2 = \| \sum\limits_{i=n+1}^{\infty} B_i \overline{\psi}_i(x) \|_{W_2^2}^2 = \sum\limits_{i=n+1}^{\infty} (B_i)^2$ *and* $\|r_{n-1}(x)\|_{W_2^2}^2 = \sum\limits_{i=n}^{\infty} (B_i)^2$.

Thus, $\|r_n(x)\|_{W_2^2} \leq \|r_{n-1}(x)\|_{W_2^2}$*, consequently, the error* $r_n(x)$ *is monotone decreasing in sense of* the norm of $W_2^2[a,b]$.

The approximate solution $u_n^N(x)$ can be obtained by taking finitely many terms in the series representation of $u_n(x)$, and

$$u_n^N(x) = \sum_{i=1}^{N} \sum_{k=1}^{i} \beta_{ik} F(x_k, u_{n-1}(x_k), Tu_{n-1}(x_k)) \overline{\psi}_i(x). \tag{2.18}$$

2.1.4 Numerical Experiment

In order to have a clear overview of our method, some numerical examples are studied to demonstrate the accuracy of the present method. Results obtained by the method are compared with the analytical solution of each example and are found to be in good agreement with each other.

Example 2.1.1 (Shahmorad, 2005) *Consider the linear Volterra IDE*

$$u'(x) = 1 - \int_0^x u(t)dt,\ 0 \le x \le 1;$$
$$u(0) = 0.$$

The exact solution is $u(x) = \sin(x)$.

Using RKHS method, taking $x_i = \frac{i-1}{n-1}$, $i = 1, 2, ..., n$ *with the reproducing kernel* $K_x(y)$ *on* $[0,1]$, *the approximate solutions* $u_n(x)$ *are calculated by Equation* (2.12). *The numerical results at some selected grid points for* $n = 51$ *and* $n = 101$ *are given in Table* 2.1 *and Table* 2.2, *respectively.*

Furthermore, the absolute errors $|u_n(x) - u(x)|$ *between the approximate solution and exact solution are shown in Figure 3.1. This illustrates that the accuracy of approximate solution will be getting better and better as* n *increases.*

Table 2.1: Numerical results for Example 2.1.1 ($n = 51$).

x	*Exact Solution* $u(x)$	*Approximate Solution*	*Absolute Error*
0.0	0.000000	0.000000	$0.0000E-00$
0.1	0.099833	0.099830	3.3278×10^{-6}
0.2	0.198669	0.198663	6.62236×10^{-6}
0.3	0.295520	0.295510	9.85074×10^{-6}
0.4	0.389418	0.389405	1.29807×10^{-5}
0.5	0.479426	0.479410	1.5981×10^{-5}
0.6	0.564642	0.564624	1.88215×10^{-5}
0.7	0.644218	0.644196	2.14741×10^{-5}
0.8	0.717356	0.717332	2.3912×10^{-5}
0.9	0.783327	0.783301	2.61111×10^{-5}
1	0.841471	0.841443	2.80492×10^{-5}

Table 2.2: Numerical results for Example 2.1.1 ($n = 101$).

x	*Exact Solution* $u(x)$	*Approximate Solution*	*Absolute Error*
0.0	0	0	$0.0000E-00$
0.1	0.099833	0.0998326	8.31947×10^{-7}
0.2	0.198669	0.198668	1.65558×10^{-6}
0.3	0.29552	0.295518	2.46267×10^{-6}
0.4	0.389418	0.389415	3.24516×10^{-6}
0.5	0.479426	0.479422	3.99522×10^{-6}
0.6	0.564642	0.564638	4.70536×10^{-6}
0.7	0.644218	0.644212	5.36849×10^{-6}
0.8	0.717356	0.71735	5.97798×10^{-6}
0.9	0.783327	0.78332	6.52774×10^{-6}
1	0.841471	0.841464	7.01227×10^{-6}

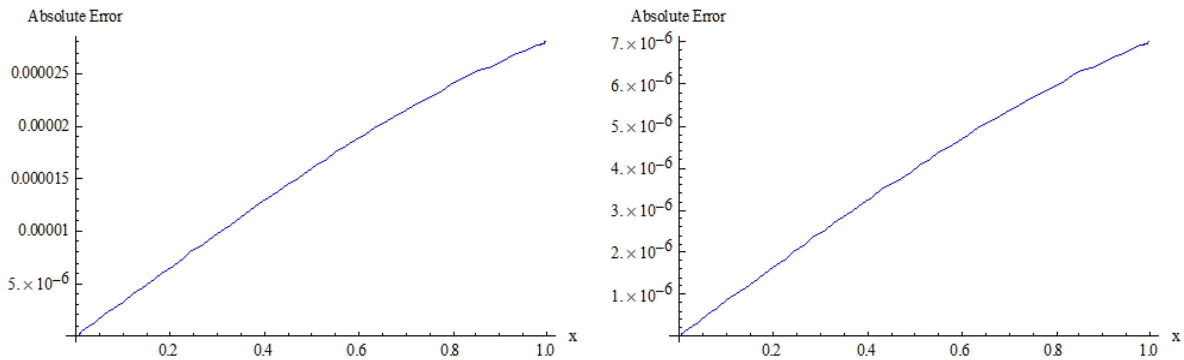


Fig. 3.1. Absolute errors for Example 2.1.1, n=51, 101.

The left: n=51; the right: n=101.

Example 2.1.2 (Zarebnia, 2010) *Consider the linear Volterra IDE*

$$u'(x) = \frac{1}{1+x} - \ln(x+1)\left(\frac{1}{2}x\ln(x+1)+1\right) + u(t) + \int_0^x \frac{x}{t+1}u(t)dt,\ 0 \le x \le 1,$$

$$u(0) = 0.$$

The exact solution is $u(x) = \ln(x+1)$.

Using RKHS method, taking $x_i = \frac{i-1}{n-1}$, $i = 1, 2, ..., n$ *with the reproducing kernel* $R_x(y)$ *on* $[0,1]$, *the approximate solutions* $u_n(x)$ *are calculated by Equation* (2.12). *The numerical results for* $n = 101$ *are given in Table* 2.3.

Table 2.3: Numerical results for Example 2.1.2.

x	*Exact Solution*	*Numerical Solution*	*Absolute Error*	*Relative Error*
0.0	0	0	$0.0000E-00$	*Indeterminate*
0.1	0.0953102	0.095311	8.68632×10^{-7}	9.11373×10^{-6}
0.2	0.182322	0.182323	1.39101×10^{-6}	7.62942×10^{-6}
0.3	0.262364	0.262366	1.66944×10^{-6}	6.36308×10^{-6}
0.4	0.336472	0.336474	1.77108×10^{-6}	5.26367×10^{-6}
0.5	0.405465	0.405467	1.74145×10^{-6}	4.29494×10^{-6}
0.6	0.470004	0.470005	1.61232×10^{-6}	3.43043×10^{-6}
0.7	0.530628	0.53063	1.40639×10^{-6}	2.65043×10^{-6}
0.8	0.587787	0.587788	1.14026×10^{-6}	1.93992×10^{-6}
0.9	0.641854	0.641855	8.26246×10^{-7}	1.28728×10^{-6}
1	0.693147	0.693148	4.73691×10^{-7}	6.83392×10^{-7}

Example 2.1.3 (Bahuguna, et al., 2009) *Consider the nonlinear Volterra IDE*

$$u'(x) = -1 + \int_0^x u^2(t)dt,\ 0 \le x \le 1,$$

$$u(0) = 0.$$

The exact solution is $u(x) = \left(\frac{1}{28}x^4 - x\right) / \left(\frac{1}{21}x^3 + 1\right)$.

Using RKHS method, taking $x_i = \frac{i-1}{n-1}$, $i = 1, 2, ..., n$ *with the reproducing kernel* $K_x(y)$ *on* $[0, 1]$, *the approximate solutions* $u_n^N(x)$ *are calculated by Equation* (2.18). *The numerical results for* $n = 51$ *with* $N = 5$ *are given in Table* 2.4.

Table 2.4: Numerical results for Example 2.1.3.

x	*Exact Solution*	*Numerical Solution*	*Absolute Error*	*Relative Error*
0.0	0.0000000	0.0000000	$0.0000E - 00$	*Indeterminate*
0.1	-0.0999917	-0.0999913	3.33242×10^{-7}	-3.3327×10^{-6}
0.2	-0.199867	-0.199865	1.33047×10^{-6}	-6.6568×10^{-6}
0.3	-0.299326	-0.299323	2.97841×10^{-6}	-9.95038×10^{-6}
0.4	-0.397873	-0.397868	5.24281×10^{-6}	-1.31771×10^{-5}
0.5	-0.494822	-0.494814	8.0591×10^{-6}	-1.62868×10^{-5}
0.6	-0.58931	-0.589299	1.13246×10^{-5}	-1.92167×10^{-5}
0.7	-0.680313	-0.680298	1.48938×10^{-5}	-2.18926×10^{-5}
0.8	-0.766679	-0.76666	1.85778×10^{-5}	-2.42315×10^{-5}
0.9	-0.847159	-0.847137	2.21478×10^{-5}	-2.61436×10^{-5}
1	-0.920455	-0.920429	2.53429×10^{-5}	-2.7533×10^{-5}

Example 2.1.4 (Bahuguna, et al., 2009) *Consider the nonlinear Volterra IDE*

$$u'(x) = 1 + \int_0^x u'(t)\, u(t) dt, 0 \leq x \leq 1,$$
$$u(0) = 0.$$

The exact solution is $u(x) = \sqrt{2} \tan\left(\frac{1}{\sqrt{2}}x\right)$.

Using RKHS method, taking $x_i = \frac{i-1}{n-1}$, $i = 1, 2, ..., n$ *with the reproducing kernel* $R_x(y)$ *on* $[0, 1]$, *the approximate solutions* $u_n^N(x)$ *are calculated by Equation* (2.18). *The numerical results for* $n = 51$ *with* $N = 5$ *are given in Table* 2.5.

Table 2.5: Numerical results for Example 2.1.4.

x	*Exact Solution*	*Numerical Solution*	*Absolute Error*	*Relative Error*
0.0	0.000000	0.000000	$0.0000E-00$	Indeterminate
0.1	0.100167	0.100166	6.74803×10^{-7}	6.73678×10^{-6}
0.2	0.201344	0.201343	1.22608×10^{-6}	6.08948×10^{-6}
0.3	0.304583	0.304581	1.51357×10^{-6}	4.96932×10^{-6}
0.4	0.411019	0.411018	1.36211×10^{-6}	3.31398×10^{-6}
0.5	0.521931	0.52193	5.51474×10^{-7}	1.05661×10^{-6}
0.6	0.638796	0.638797	1.12636×10^{-6}	1.76326×10^{-6}
0.7	0.763386	0.763389	3.51502×10^{-6}	4.60451×10^{-6}
0.8	0.897882	0.897886	4.74822×10^{-6}	5.28825×10^{-6}
0.9	1.04504	1.04504	2.99036×10^{-6}	2.86147×10^{-6}
1	1.20846	1.20841	4.54341×10^{-5}	3.75967×10^{-5}

Example 2.1.5 (Babolian, et al., 2009) *Consider the nonlinear Volterra IDE*

$$u'(x) = 2\sin(x)\cos(x) - \int_0^x 3\cos(x-t)\,u^2(t)dt, 0 \le x \le 1,$$

$$u(0) = 1.$$

The exact solution is $u(x) = \cos(x)$.

Using RKHS method, taking $x_i = \frac{i-1}{n-1}$, $i = 1, 2, ..., n$ *with the reproducing kernel* $K_x(y)$ *on* $[0, 1]$, *the approximate solutions* $u_n(x)$ *are calculated by Equation* (2.14). *Table* 2.6. *shows the numerical results for* $n = 11$ *and comparison with the other numerical solutions.*

Table 2.6: Numerical results for Example 2.1.5.

x	Exact sol.	Presented M.	Method [7]	BPFs method [29]	Adomian's M. [30]
0.0	1.000000	1.000000	1.000000	1.000000	1.000000
0.1	0.995004	0.995058	0.994555	0.995141	0.994951
0.2	0.980067	0.98028	0.979825	0.975784	0.980303
0.3	0.955336	0.955796	0.955174	0.960386	0.955685
0.4	0.921061	0.921795	0.920861	0.918443	0.921165
0.5	0.877583	0.878479	0.877921	0.862193	0.877048
0.6	0.825336	0.826005	0.825397	0.828963	0.822596
0.7	0.764842	0.764416	0.765164	0.752929	0.755333
0.8	0.696707	0.693569	0.697142	0.710418	0.667739
0.9	0.621610	0.613055	0.622057	0.617232	0.547241
1	0.540302	0.522129	0.541102	0.566917	0.364798

2.2 Fredholm Integro-Differential Equation

This section investigates the numerical solutions of the first-order Fredholm IDEs using the reproducing kernel method. It is a relatively new analytical technique. The analytical solution $u(x)$ and approximate solution $u_n(x)$ are represented in the form of series in $W_2^2[a,b]$.

2.2.1 Introduction

We consider the Fredholm IDE of the following form:

$$\begin{aligned} u'(x) &= F(x, u(x), Tu(x)), \ \ a \le x \le b; \\ u(a) &= u_0. \end{aligned} \tag{2.19}$$

where $Tu(x) = \int_a^b h(x,t)u(t)\,dt$, a, b and u_0 are real finite constants, $h(x,t)$ is known continuous function, $u(x)$ is unknown function to be determined and $F(x, t, u(t), Tu(x))$

is a linear or nonlinear function depending on the problem discussed. We suppose that IDE (2.19) have a unique solution.

In the same manner, we construct the reproducing kernel spaces $W_2^1[a,b]$ and $W_2^2[a,b]$. After homogenization of the initial condition, the IDE (2.19) can be converted into the equivalent form:

$$\begin{cases} Lu(x) = F(x, u(x), Tu(x)), \ a \leq x \leq b; \\ u(a) = 0. \end{cases} \tag{2.20}$$

where $x \in [a,b]$, $u(x) \in W_2^2[a,b]$ and $F(x, u(x), Tu(x)) \in W_2^1[a,b]$.

$L : W_2^2[a,b] \rightarrow W_2^1[a,b]$ is a bounded linear operator such that $Lu(x) = u'(x)$. A complete normal orthogonal system $\{\overline{\psi}_i(x)\}_{i=1}^{\infty}$ is given by $\overline{\psi}_i(x) = \sum_{k=1}^{i} \beta_{ik}\psi_k(x)$, where β_{ik} are coefficients of Gram-Schmidt orthonormalization.

The analytical solution $u(x)$ and approximate solution $u_n(x)$ are represented in the form of series in $W_2^2[a,b]$. That means, the analytical solution $u(x)$ and approximate solution $u_n(x)$ are given respectively by

$$u(x) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \beta_{ik} [F(x_k, u(x_k), Tu(x_k))] \overline{\psi}_i(x), \tag{2.21}$$

$$u_n(x) = \sum_{i=1}^{n} \sum_{k=1}^{i} \beta_{ik} [F(x_k, u(x_k), Tu(x_k))] \overline{\psi}_i(x), \tag{2.22}$$

Furthermore, Theorems .2.1.1 and .2.1.2 hold. Thus, if Equation (2.19) is nonlinear, the approximate solution $u_n^N(x)$ can be obtained by taking finitely many terms in the series representation of $u_n(x)$, using the iterative method, and

$$u_n^N(x) = \sum_{i=1}^{N} \sum_{k=1}^{i} \beta_{ik} F(x_k, u_{n-1}(x_k), Tu_{n-1}(x_k)) \overline{\psi}_i(x). \tag{2.23}$$

2.2.2 Numerical Experiment

In this section, some numerical examples are given to clarify the high accuracy of the method. The results of examples show that our method is very useful in computing a numerical solution of nonlinear IDE.

Example 2.2.1 (Darania, et al., 2007) *Consider the linear Fredholm IDE*

$$u'(x) = 1 - \tfrac{1}{3}x + \int_0^1 xtu(t)dt,\ 0 \le x \le 1;$$
$$u(0) = 0.$$

The exact solution is $u(x) = x$.

Using RKHS method, taking $x_i = \frac{i-1}{n-1}$, $i = 1, 2, ..., n$ *with the reproducing kernel* $K_x(y)$ *on* $[0,1]$, *the approximate solutions* $u_n(x)$ *are calculated by Equation* (2.22). *The numerical results for* $n = 26$ *are given in Table* 2.7.

Moreover, Table 2.8 *shows the numerical results for Example* 2.2.1 *for* $n = 26$ *which include the absolute error at some selected nodes in* $[0,1]$ *for* $u'(x)$.

Table 2.7: Numerical results for Example 2.2.1.

x	*Analytical Solution*	*Absolute Error*	*Relative Error*
0.16	0.16	2.77556×10^{-17}	1.73472×10^{-16}
0.32	0.32	1.11022×10^{-16}	3.46945×10^{-16}
0.48	0.48	2.22045×10^{-16}	4.62593×10^{-16}
0.64	0.64	4.44089×10^{-16}	6.93889×10^{-16}
0.80	0.80	4.44089×10^{-16}	5.55112×10^{-16}
0.96	0.96	6.66134×10^{-16}	6.93889×10^{-16}

Table 2.8: Numerical results for $u'(x)$, Example 2.2.1.

x	*Analytical Solution*	*Numerical Solution*	*Absolute Error*	*Relative Error*
0.16	1	0.9999999999999997	3.33067×10^{-16}	3.33067×10^{-16}
0.32	1	0.9999999999999998	2.22045×10^{-16}	2.22045×10^{-16}
0.48	1	0.9999999999999992	7.77156×10^{-16}	7.77156×10^{-16}
0.64	1	0.9999999999999990	9.99201×10^{-16}	9.99201×10^{-16}
0.80	1	0.9999999999999988	1.22125×10^{-15}	1.22125×10^{-15}
0.96	1	0.9999999999999992	7.77156×10^{-16}	7.77156×10^{-16}

Example 2.2.2 (Darania, et al., 2007) *Consider the linear Fredholm IDE*

$$u'(x) = xe^x + e^x - x + \int_0^1 xu(t)dt,\ 0 \le x \le 1;$$
$$u(0) = 0.$$

The exact solution is $u(x) = xe^x$.

Using RKHS method, taking $x_i = \frac{i-1}{n-1}$, $i = 1, 2, ..., n$ *with the reproducing kernel* $K_x(y)$ *on* $[0, 1]$, *the approximate solutions* $u_n(x)$ *are calculated by Equation* (2.22). *The numerical results for* $n = 11$ *are given in Table* 2.9.

Example 2.2.3 (Ghasemi, et al., 2007) *Consider the nonlinear Fredholm IDE*

$$u'(x) = 1 - e^{-1} + \int_0^1 e^{-u'(t)}dt,\ 0 \le x \le 1;$$
$$u(0) = 0.$$

The exact solution is $u(x) = x$.

Using RKHS method, taking $x_i = \frac{i-1}{n-1}$, $i = 1, 2, ..., n$ *with the reproducing kernel* $K_x(y)$ *on* $[0, 1]$, *the approximate solutions* $u_n^N(x)$ *are calculated by Equation* (2.23). *The numerical results for* $n = 11$ *with* $N = 5$ *are given in Table* 2.10.

Table 2.9: Numerical results for Example 2.2.2.

x	Exact sol.	Numerical sol.	Presented method	Method in [24]	Method in [40]
0.0	0.0	0.0	$0.0000E-00$	$0.0000E-00$	$0.0000E-00$
0.1	0.110517	0.110366	1.50879×10^{-4}	1.001183×10^{-2}	$1.34917637 \times 10^{-3}$
0.2	0.244281	0.244068	2.12891×10^{-4}	2.786514×10^{-2}	$1.15960044 \times 10^{-3}$
0.3	0.404958	0.404777	1.8104×10^{-4}	5.087309×10^{-2}	$5.67152531 \times 10^{-3}$
0.4	0.59673	0.59668	4.97023×10^{-5}	7.553563×10^{-2}	$5.93105645 \times 10^{-2}$
0.5	0.824361	0.824548	1.8746×10^{-4}	9.718886×10^{-2}	$1.32330751 \times 10^{-2}$
0.6	1.09327	1.09381	5.37579×10^{-4}	1.095517×10^{-1}	$4.39287720 \times 10^{-2}$
0.7	1.40963	1.41064	1.00868×10^{-3}	1.041332×10^{-1}	$1.41201624 \times 10^{-2}$
0.8	1.78043	1.78204	1.6098×10^{-3}	0.945127×10^{-2}	$1.34514117 \times 10^{-2}$
0.9	2.21364	2.21599	2.3511×10^{-3}	1.000343×10^{-2}	$1.32045209 \times 10^{-2}$
1	2.71828	2.72153	3.244×10^{-3}	1.551477×10^{-1}	$---$

Table 2.10: Numerical results for Example 2.2.3.

x	*Exact Solution*	*Numerical Solution*	*Absolute Error*	*Relative Error*
0.16	0.16	0.16	2.77556×10^{-17}	1.73472×10^{-16}
0.32	0.32	0.32	5.55112×10^{-17}	1.73472×10^{-16}
0.48	0.48	0.48	5.55112×10^{-16}	1.15648×10^{-16}
0.64	0.64	0.64	2.22045×10^{-16}	3.46945×10^{-16}
0.8	0.8	0.8	1.11022×10^{-16}	1.38778×10^{-16}
0.96	0.96	0.96	3.33067×10^{-16}	3.46945×10^{-16}

Example 2.2.4 (Darania, et al., 2007) *Consider the nonlinear Fredholm IDE*

$$\begin{aligned} &u'(x) = 1 - \tfrac{1}{3}x^3 + \int_0^1 x^3 u^2(t)\,dt, \; 0 \le x \le 1; \\ &u(0) = 0. \end{aligned} \tag{2.32}$$

The exact solution is $u(x) = x$.

Using RKHS method, taking $x_i = \frac{i-1}{n-1}$, $i = 1, 2, ..., n$ *with the reproducing kernel* $K_x(y)$ *on* $[0,1]$, *the approximate solutions* $u_n^N(x)$ *are calculated by Equation* (2.23). *The numerical results for* $n = 11$ *with* $N = 5$ *are given in Table* 2.11.

Table 2.11. Numerical results for Example 2.2.4.

x	*Exact Solution*	*Numerical Solution*	*Absolute Error*	*Relative Error*
0.0	0.0	0.0	$0.0000E-00$	*Indeterminate*
0.1	0.1	0.1	8.27199×10^{-10}	8.27199×10^{-9}
0.2	0.2	0.2	8.27199×10^{-9}	4.13599×10^{-8}
0.3	0.3	0.3	3.72239×10^{-8}	1.2408×10^{-7}
0.4	0.4	0.4	1.12499×10^{-7}	2.81248×10^{-7}
0.5	0.5	0.5	2.6884×10^{-7}	5.37679×10^{-7}
0.6	0.6	0.599999	5.50914×10^{-7}	9.18191×10^{-7}
0.7	0.7	0.699999	1.01332×10^{-6}	1.4476×10^{-6}
0.8	0.8	0.799998	1.72057×10^{-6}	2.15072×10^{-6}
0.9	0.9	0.899997	2.74713×10^{-6}	3.05236×10^{-6}
1	1	0.999996	4.17735×10^{-6}	4.17735×10^{-6}

Chapter Three

Second-Order Integro-Differential Equation

In this chapter, we apply the RKHS technique to develop a novel numerical method in the space $W_2^3[a,b]$ for obtaining approximations to the solution and its derivatives for a general form of second-order Volterra IDEs. It is a relatively new analytical technique. The analytical solution $u(x)$ and approximate solution $u_n(x)$ are represented in terms of series in the space $W_2^3[a,b]$. The n-term approximation is obtained and is proved to converge to the analytical solution. Moreover, implementations of the method on a nonlinear Volterra IDE of the following type

$$u''(x) = F(x, u(x), u'(x), Tu(x))$$

will be given in $W_2^3[a,b]$, where $Tu(x) = \int_a^x G(x,t,u(t))\,dt$.

Furthermore, two numerical examples are studied to demonstrate the accuracy of the present method. Results obtained by the method are compared with the analytical solution of each example, are found to be in good agreement with each other and are very useful in computing a numerical solution of nonlinear Volterra IDE.

3.1 Introduction

Consider the second-order Volterra IDE of the following form:

$$\begin{aligned} &u''(x) = F(x, u(x), u'(x), Tu(x)),\ \ a \le x \le b; \\ &u(a) = \alpha,\ \ u'(a) = \beta. \end{aligned} \tag{3.1}$$

where $Tu(x) = \int_a^b h(x,t)u(t)\,dt$, a, b and α, β are real finite constants, $h(x,t)$ is known continuous function, $F(x, u(x), u'(x), Tu(x))$ is a linear or nonlinear function depend-

ing on the problem discussed, and $u(x)$ is unknown function to be determined. We suppose that IDE (3.1) have a unique solution.

The advantages of this method are as follows: Firstly, the conditions for determining solutions in Equations (3.1) can be imposed on the reproducing kernel space. Secondly, the conditions about the nonlinearity F are simple and F is allowed to include not only x, $u(x)$, $u'(x)$, but also $Tu(x)$. Thirdly, the iterative sequence $u_n(x)$ of approximate solutions converges in C^2 to the solution $u(x)$.

For the solution of the above Volterra IDE, we give two reproducing kernel spaces.

(i) The space $W_2^1[a,b]$ is defined by $W_2^1[a,b] = \{u : u \text{ is Abs.C}, u' \in L^2[a,b]\}$. The inner product and the norm in the space $W_2^1[a,b]$ are given, respectively, by

$$\langle u, v\rangle_{W_2^1} = \int_a^b u(t)\, v(t) + u'(t)\, v'(t)\, dt, \tag{3.2}$$

and $\|u\|_{W_2^1} = \sqrt{\langle u, u\rangle_{W_2^1}}$ where $u, v \in W_2^1[a,b]$.

In (Cui and Li, 2003), it has been proved that the space $W_2^1[a,b]$ is a complete RKHS and its reproducing kernel is given by

$$T_x(y) = \frac{1}{2\sinh(b-a)}\left[\cosh(x+y-b-a) + \cosh(|x-y| - b + a)\right].$$

(ii) The space $W_2^3[a,b]$ is defined by

$$W_2^3[a,b] = \{u(x) \mid u, u', u'' \text{ are Abs.C real-valued functions, } u^{(3)} \in L^2[a,b],$$
$$u(a) = u'(a) = 0\} \tag{3.3}$$

Note that the space $W_2^3[a,b]$ is a Hilbert space. For $u, v \in W_2^3[a,b]$, the inner product and the norm in the space $W_2^3[a,b]$ are given by

$$\langle u, v\rangle_{W_2^3} = \sum_{i=0}^{2} u^{(i)}(a)\, v^{(i)}(a) + \int_a^b u^{(3)}(y) v^{(3)}(y) dy \tag{3.4}$$

and $||u||_{W_2^3} = \sqrt{\langle u, u\rangle_{W_2^3}}$, respectively.

From the definition of the reproducing kernel spaces $W_2^1[a,b]$ and $W_2^3[a,b]$, we get $W_2^1[a,b] \supset W_2^3[a,b]$, for any $u \in W_2^3[a,b]$, and $\|u\|_{W_2^1} \leq \|u\|_{W_2^3}$.

Theorem 3.1.1 *The space $W_2^3[a,b]$ is a reproducing kernel space. That is, for each fixed $x \in [a,b]$ and any $u(y) \in W_2^3[a,b]$, there exist $K_x(y) \in W_2^3[a,b]$, $y \in [a,b]$ such that $\langle u(y), K_x(y)\rangle_{W_2^3} = u(x)$, and the reproducing kernel $K_x(y)$ can be presented by*

$$K_x(y) = \begin{cases} p_1(x) + p_2(x)y + p_3(x)y^2 + p_4(x)y^3 + p_5(x)y^4 + p_6(x)y^5, & y \leq x; \\ q_1(x) + q_2(x)y + q_3(x)y^2 + q_4(x)y^3 + q_5(x)y^4 + q_5(x)y^5, & y > x. \end{cases} \tag{3.5}$$

Proof. *Firstly, we show that the space $W_2^3[a,b]$ is RKHS. Afterward, we find out the expression form of the reproducing kernel function $K_x(y)$ in the space $W_2^3[a,b]$.*

Using the fact

$$\sum_{i=0}^{2} (-1)^i u^{(2-i)}(y) K_x^{(3+i)}(y) = \sum_{i=0}^{2} (-1)^{2-i} u^{(i)}(y) K_x^{(5-i)}(y)$$

and through several integrations by parts for Equation (3.4), we have

$$\int_a^b u^{(3)}(y) K_x^{(3)}(y)\,dy = \sum_{i=0}^{2} (-1)^{2-i} u^{(i)}(y) K_x^{(5-i)}(y)\Big|_{y=a}^{y=b} + (-1)^3 \int_a^b u(y) K_x^{(6)}(y)\,dy.$$

Thus, Equation (3.4) becomes

$$\begin{aligned} \langle u(y), K_x(y)\rangle_{W_2^3} &= \sum_{i=0}^{2} u^{(i)}(a)\left[K_x^{(i)}(a) - (-1)^{2-i} K_x^{(5-i)}(a)\right] \\ &\quad + \sum_{i=0}^{2} (-1)^{2-i} u^{(i)}(b) K_x^{(5-i)}(b) + (-1)^3 \int_a^b u(y) K_x^{(6)}(y)\,dy. \end{aligned}$$

Since $K_x(y) \in W_2^3[a,b]$, it follows that

$$K_x(a) = K_x'(a) = 0. \tag{3.6}$$

Again, for $u \in W_2^3[a,b]$, *we have* $u(a)=0,\ u'(a)=0$. *Hence, we get the following differential equations:*

$$\begin{cases} K_x^{(2)}(a) - K_x^{(3)}(a) = 0, \\ K_x^{(i)}(b) = 0,\ i = 3,4,5. \end{cases} \tag{3.7}$$

Which implies that

$$\langle u(y), K_x(y)\rangle_{W_2^3} = \int_a^b u(y)(-K_x^{(6)}(y))dy$$

Now, for each $x \in [a,b]$, *if* $K_x(y)$ *also satisfy*

$$-K_x^{(6)}(y) = \delta(x-y), \tag{3.8}$$

where δ *is the dirac-delta function, then* $\langle u(y), K_x(y)\rangle_{W_2^3} = u(x)$. *Obviously,* $K_x(y)$ *is the reproducing kernel of the space* $W_2^3[a,b]$.

In the following, we will get the expression of the reproducing kernel $K_x(y)$. *Note that the characteristic equation of Equation* (3.8) *is given by* $\lambda^6 = 0$, *and their characteristic values are* $\lambda = 0$ *with* 6 *multiple roots. So, we present the reproducing kernel* $K_x(y)$ *by*

$$K_x(y) = \begin{cases} p_1(x) + p_2(x)y + p_3(x)y^2 + p_4(x)y^3 + p_5(x)y^4 + p_6(x)y^5, & y \le x; \\ q_1(x) + q_2(x)y + q_3(x)y^2 + q_4(x)y^3 + q_5(x)y^4 + q_5(x)y^5, & y > x. \end{cases}$$

On the other hand, for $K_x(y) \in W_2^3[a,b]$, *let* $K_x(y)$ *satisfy*

$$K_x^{(m)}(x+0) = K_x^{(m)}(x-0),\ m = 0,1,...,4. \tag{3.9}$$

Integrating Equation (3.8) *from* $x-\varepsilon$ *to* $x+\varepsilon$ *with respect to* y *and let* $\varepsilon \to 0$, *we have*

the jump degree of $K_x^{(5)}(y)$ *at* $y = x$ *given by*

$$K_x^{(5)}(x-0) - K_x^{(5)}(x+0) = 1 \tag{3.10}$$

Through Equations (3.6), (3.7), (3.9), *and* (3.10) , *the unknown coefficients of Equation* (3.5) *can be obtained. This completes the proof.*

The unique representation of the reproducing kernel $K_x(y)$ is provided by

$$K_x(y) = \begin{cases} \frac{-1}{120}(a-y)^2 \begin{pmatrix} 6a^3+5xy^2-y^3-10x^2(3+y)-3a^2(10 \\ +5x+y)+2a(5x^2-y^2+5x(6+y)) \end{pmatrix}, y \le x, \\ \frac{1}{120}(a-x)^2 \begin{pmatrix} -6a^3+x^3-5x^2y+30y^2+10xy^2+3a^2(x \\ +10+5y)+2a(x^2-5x^2-5y(6+y)) \end{pmatrix}, y > x. \end{cases}$$

We mention here that the kernel $K_x(y)$ is symmetric, that is $K_x(y) = K_y(x)$ for each x and y, and $K_x(x) \ge 0$ for any fixed $x \in [a,b]$.

3.2 Numerical Method and Convergence Analysis

In this section, the existence of the solution and an iterative method of obtaining the solution of IED (3.1) are developed in the reproducing kernel space $W_2^3[a,b]$.

To do this, we define a linear operator $L : W_2^3[a,b] \to W_2^1[a,b]$, where $Lu = u''$. After homogenization of the initial conditions of Equation (3.1), the IDE (3.1) can be converted into the equivalent form as follows:

$$\begin{aligned} Lu(x) &= F(x, u(x), u'(x), Tu(x)), \\ u(a) &= \ u'(a) = 0, \end{aligned} \tag{3.11}$$

where $x \in [a,b]$, $u(x) \in W_2^3[a,b]$, and $F(x, u(x), u'(x), Tu(x)) \in W_2^1[a,b]$. It is clear that L is a bounded linear operator.

Now, we construct an orthogonal function system of $W_2^3[a,b]$. Let $\varphi_i(x) = T_{x_i}(x)$

and $\psi_i(x) = L^*\phi_i(x)$, where $\{x_i\}_{i=1}^{\infty}$ is dense in the interval $[a,b]$, and L^* is the adjoint operator of L. From the properties of $T_{x_i}(x)$, for every $u(x) \in W_2^3[a,b]$, we have $\langle u(x), \varphi_i(x)\rangle_{W_2^1} = \langle u(x), T_{x_i}(x)\rangle_{W_2^1} = u(x_i)$. Since $K_x(y) \in W_2^3[a,b]$, it follows that

$$\psi_i(x) = L^*\phi_i(x) = \langle L^*\phi_i(y), K_x(y)\rangle_{W_2^3} = \langle \phi_i(y), L_y K_x(y)\rangle_{W_2^1} = L_y K_x(y)\big|_{y=x_i},$$

where L_y indicates that the operator L applies to the function of y.

Practise Gram–Schmidt orthonomalization for $\{\psi_i(x)\}_{i=1}^{\infty}$. Then, the orthonormal system $\{\overline{\psi}_i(x)\}_{i=1}^{\infty}$ of the space $W_2^3[a,b]$ can be derived as follows:

$$\overline{\psi}_i(x) = \sum_{k=1}^{i} \beta_{ik}\psi_k(x),\ \beta_{ii} > 0,\ i = 1, 2, \ldots \tag{3.12}$$

Lemma 3.2.1 *Let $\{x_i\}_{i=1}^{\infty}$ be dense on $[a,b]$, then $\{\overline{\psi}_i(x)\}_{i=1}^{\infty}$, is the complete function system of the space $W_2^3[a,b]$.*

Proof. *For each fixed $u(x) \in W_2^3[a,b]$, if $\langle u(x), \psi_i(x)\rangle_{W_2^3} = 0$, $i = 1, 2, \ldots$, then*

$$\langle u(x), L^*\phi_i(x)\rangle_{W_2^3} = \langle Lu(x), \phi_i(x)\rangle_{W_2^1} = (Lu)(x_i) = 0.$$

Since $\{x_i\}_{i=1}^{\infty}$ is dense on the interval $[a,b]$, therefore $Lu(x) = 0$. It follows that $u(x) = 0$ from the existence of L^{-1}.

Lemma 3.2.2 *If $u(x) \in W_2^3[a,b]$, then there exists $M_i > 0$, $i = 0, 1, 2$, such that $\left|\left|u^{(i)}(x)\right|\right|_{C^2} \le M_i \left|\left|u(x)\right|\right|_{W_2^3}$.*

Proof. *By the properties of the reproducing kernel function $K_x(y)$, we have $u^{(i)}(x) = \langle\, u(y), K_x^{(i)}(y)\,\rangle_{W_2^3}$ and $\|\, K_x^{(i)}(y)\,\|_{W_2^3} \le M_i$, $i = 0, 1, 2$, for any $x, y \in [a,b]$. Then*

$$\left|u^{(i)}(x)\right| = \left|\langle\, u(x), K_x^{(i)}(x)\,\rangle_{W_2^3}\right| \le \left|\left|u^{(i)}(x)\right|\right|_{W_2^3} \|\, K_x^{(i)}(x)\,\|_{W_2^3} \le M_i \left|\left|u^{(i)}(x)\right|\right|_{W_2^3},$$

thus $\left|\left|u^{(i)}(x)\right|\right|_{C^2} \le M_i \left|\left|u(x)\right|\right|_{W_2^3}$, $i = 0, 1, 2$.

Now, we will give the representation of the exact solution of IDE (3.1) in the space $W_2^3[a,b]$. Later, the convergence of approximate solution $u_n(x)$ to the analytic solution will be proved.

Theorem 3.2.1 *If $\{x_i\}_{i=1}^{\infty}$ is dense on $[a,b]$ and the solution of IDE (3.1) is unique, then the solution of IDE (3.1) satisfies the form:*

$$u(x)=\sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}F(x_k,u(x_k),u'(x_k),Tu(x_k))\overline{\psi}_i(x). \tag{3.13}$$

Proof. *Let $u(x)$ be a solution of IDE (3.1) in the space $W_2^3[a,b]$. Since $\{\overline{\psi}_i(x)\}_{i=1}^{\infty}$ is the complete orthonormal basis of the space $W_2^3[a,b]$. Thus, $u(x)$ can be expanded in the Fourier series as*

$$u(x)=\sum_{i=1}^{\infty}\langle u(x),\overline{\psi}_i(x)\rangle_{W_2^3}\overline{\psi}_i(x).$$

From Equation (3.12) and since $\langle u(x),\phi_i(x)\rangle_{W_2^3}=u_i(x)$ for each $u(x)\in W_2^1[a,b]$, we have

$$\begin{aligned}
u(x) &= \sum_{i=1}^{\infty}\langle\, u(x),\sum_{k=1}^{i}\beta_{ik}\psi_k(x)\,\rangle_{W_2^3}\overline{\psi}_i(x)\\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\langle u(x),\psi_k(x)\rangle_{W_2^3}\overline{\psi}_i(x)\\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\langle u(x),L^*\phi_k(x)\rangle_{W_2^3}\overline{\psi}_i(x)\\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\langle Lu(x),\phi_k(x)\rangle_{W_2^1}\overline{\psi}_i(x)\\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\langle F(x,u(x),u'(x),Tu(x)),\phi_k(x)\rangle_{W_2^1}\overline{\psi}_i(x)\\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}F(x_k,u(x_k),u'(x_k),Tu(x_k))\overline{\psi}_i(x).
\end{aligned}$$

Corollary 3.2.1 *For any $u(x)\in W_2^3[a,b]$, $u_n^{(i)}(x)$ are uniformly convergent to $u^{(i)}(x)$, $i=0,1,2$.*

Proof. *From Lemma 3.2.2, for any* $x \in [a,b]$, *we get*

$$\begin{aligned} |u(x) - u_n(x)| &= \left| \langle u(x) - u_n(x), K_x(x) \rangle_{W_2^3} \right| \leq \|K_x(x)\|_{W_2^3} \|u(x) - u_n(x)\|_{W_2^3} \\ &\leq M_0 \|u(x) - u_n(x)\|_{W_2^3} \longrightarrow 0, \ \text{as } n \to \infty, \end{aligned}$$

and, for $i = 1, 2$, *we have*

$$\begin{aligned} \left|u^{(i)}(x) - u_n^{(i)}(x)\right| &= \left| \left\langle u(x) - u_n(x), K_x^{(i)}(x) \right\rangle_{W_2^3} \right| \leq \left\|K_x^{(i)}(x)\right\|_{W_2^3} \|u(x) - u_n(x)\|_{W_2^3} \\ &\leq M_i \|u(x) - u_n(x)\|_{W_2^3} \longrightarrow 0, \ \text{as } n \to \infty. \end{aligned}$$

Hence, $\left|u^{(i)}(x) - u_n^{(i)}(x)\right| \leq \left\|u^{(i)}(x) - u_n^{(i)}(x)\right\|_C \leq M_i \left\|u^{(i)}(x) - u_n^{(i)}(x)\right\|_{W_2^3} \longrightarrow 0$, $i = 0, 1, 2$. *Thus, the approximate solution* $u_n^{(i)}(x)$ *converge uniformly to* $u^{(i)}(x)$,$i = 0, 1, 2$.

Remark 3.2.1 *If Equation* (3.1) *is linear, then the* n-term *approximation solution* $u_n(x)$ *of* $u(x)$ *can be obtained directly by the following equation:*

$$u_n(x) = \sum_{i=1}^{n} \sum_{k=1}^{i} \beta_{ik} F(x_k, u(x_k), u'(x_k), Tu(x_k)) \overline{\psi}_i(x). \tag{3.14}$$

If Equation (3.1) *is nonlinear, the analytical and approximate solutions can be obtained using the iterative method that described in Section 2.1.3.*

Hence, the approximate solution $u_n^N(x)$ can be obtained by taking finitely many terms in the series representation of $u_n(x)$, using the iterative method, and

$$u_n^N(x) = \sum_{i=1}^{N} \sum_{k=1}^{i} \beta_{ik} F\left(x_k, u_{n-1}(x_k), u'_{n-1}(x_k), Tu_{n-1}(x_k)\right) \overline{\psi}_i(x). \tag{3.15}$$

3.3 Numerical Experiment

In this section, two numerical examples are studied to demonstrate the accuracy of the present method. Results obtained by the method are compared with the analytical

solution of each example, are found to be in good agreement with each other, and are very useful in computing a numerical solution of second-order nonlinear Volterra IDE.

We used the reproducing kernel function $K_x(x)$ on $[0,1]$ as following

$$K_x(y) = \begin{cases} \frac{1}{120} y^2 \left(5xy^2 - y^3 - 10x^2(3+y)\right), & y \leq x, \\ \frac{1}{120} x^2 \left(-x^3 + 5x^2 y - 30y^2 - 10xy^2\right), & y > x. \end{cases} \tag{3.16}$$

Example 3.3.1 (Saeidy, et al., 2010) *Consider the linear Volterra IDE*

$$u''(x) + xu(x) = f(x) + \int_0^x x^2 e^t u(t)dt, \; x \in [0,1],$$
$$u(0) = 1, \; u'(0) = 0,$$

where $f(x) = -(1+x)\cos x - \frac{1}{2}x^2 \left(e^x(\cos x + \sin x) - 1\right),$ *with the exact solution* $u(x) = \cos x.$

Using RKHS method, taking $x_i = \frac{i-1}{n-1}$, $i = 1, 2, ..., n,$ *with the reproducing kernel function that given in Equation* (3.16). *The approximate solutions* $u_n(x)$ *are calculated by Equation* (3.14). *The numerical results for* $n = 26$ *are given in Table* 3.1.

Example 3.3.2 (Saberi-Nadjafi, 2009) *Consider the nonlinear Volterra IDE*

$$u''(x) = f(x) + \int_0^x u^2(t)dt, \; x \in [0,1],$$
$$u(0) = 0, \; u'(0) = 1,$$

where $f(x) = \sinh x + \frac{1}{2}\cosh x \sinh x - \frac{1}{2}x,$ *with the exact solution* $u(x) = \sinh x.$

Using RKHS method, taking $x_i = \frac{i-1}{n-1}$, $i = 1, 2, ..., n,$ *with the reproducing kernel function that given in Equation* (3.16). *The approximate solutions* $u_n(x)$ *are calculated by Equation* (3.15). *The numerical results for* $n = 26$ *with* $N = 5$ *are given in Table* 3.2.

Table 3.1: Numerical results for Example 3.3.1.

x	*Analytical Solution*	*Numerical Solution*	*Absolute Error*	*Relative Error*
0.0	1	1	$0.0000E-00$	$0.0000E-00$
0.1	0.995004	0.995005	3.92599×10^{-7}	3.9457×10^{-7}
0.2	0.980067	0.980066	6.43854×10^{-7}	6.5695×10^{-7}
0.3	0.955336	0.955321	1.54431×10^{-5}	1.6165×10^{-5}
0.4	0.921061	0.920984	7.68699×10^{-5}	8.3458×10^{-5}
0.5	0.877583	0.877334	2.48866×10^{-4}	2.8358×10^{-4}
0.6	0.825336	0.824701	6.34858×10^{-4}	7.6921×10^{-4}
0.7	0.764842	0.763461	1.38149×10^{-3}	1.8062×10^{-3}
0.8	0.696707	0.694034	2.67303×10^{-3}	3.8367×10^{-3}
0.9	0.62161	0.616903	4.70733×10^{-3}	7.5728×10^{-3}
1	0.540302	0.532705	7.59715×10^{-3}	1.4061×10^{-2}

Table 3.2: Numerical results for Example 3.3.2.

x	*Analytical Solution*	*Numerical Solution*	*Absolute Error*	*Relative Error*
0.0	0.000000	0.000000	$0.0000E-00$	*Indeterminate*
0.1	0.100167	0.100167	2.29896×10^{-8}	2.29513×10^{-7}
0.2	0.201336	0.201336	1.80989×10^{-7}	8.9894×10^{-7}
0.3	0.30452	0.304521	6.0494×10^{-7}	1.98654×10^{-6}
0.4	0.410752	0.410754	1.43908×10^{-6}	3.50352×10^{-6}
0.5	0.521095	0.521098	2.81483×10^{-6}	5.40175×10^{-6}
0.6	0.636654	0.636658	4.88982×10^{-6}	7.68051×10^{-6}
0.7	0.758584	0.758592	7.7982×10^{-6}	1.02799×10^{-5}
0.8	0.888106	0.888118	1.17123×10^{-5}	1.31879×10^{-5}
0.9	1.02652	1.02653	1.67665×10^{-5}	1.63333×10^{-5}
1	1.1752	1.17522	2.31434×10^{-5}	1.96932×10^{-5}

Chapter Four

Forth-Order Integro-Differential Equation

In this chapter, RKHS method is applied to approximate the solution of two-point BVP for fourth-order IDE in the space $W_2^5[a,b]$. It is a relatively new analytical technique. The analytical solution $u(x)$ and approximate solution $u_n(x)$ are represented in terms of series in the space $W_2^5[a,b]$. The n-term approximation is obtained and is proved to converge to the analytical solution. Moreover, the proposed method has an advantage that it is possible to pick any point in the interval of integration and as well the approximate solutions and its all derivatives will be applicable. Numerical examples are given to demonstrate the computation efficiency of the presented method. Results obtained by the method indicate the method is simple and effective.

4.1 Introduction

BVPs for IDEs are important because they have many applications in the study of physical, biological and chemical phenomena. However, very few attempts have been made to investigate BVPs in RKHS. Recently, using the RKHS method, the authors (Yulan, et al., 2009) discussed the analytical and approximate solutions of a class of second-order IDE in the space $W_2^3[0,1]$.

In (Cui and Du, 2006, a & b), they obtained an existence theorem of solution of a class of nonlinear second-order IDE by a constructive method. In the meantime, an iterative method of obtaining the solution is presented in the space $W_2^3[0,1]$.

IDEs are usually difficult to solve analytically. So, it is required to obtain an efficient approximate solution. Therefore, they have been of great interest by several authors. In literature, there exist numerical techniques such as Wavelet-Galerkin method (Avudainayagam, et al., 2000) and (El-Sayed and Abdel-Aziz, 2003), Lagrange interpolation method (Rashed, 2004) and Tau method (Hosseini and Shahmorad, 2003); Also, there

exist semi analytical-numerical techniques such as Adomian's decomposition method (El-Sayed and Abdel-Aziz, 2003) and (Hashim, 2006), Taylor polynomials (Maleknejad and Mahmoudi, 2003), differential transform method (Arikoglu and Ozkol, 2005), variational iteration method (Sweilam, 2007) and homotopy perturbation method (Saeidy, et al., 2010). However, none of them propose a methodical way to solve these equations. Moreover, previous studies require more effort to achieve the results, they are not accurate and usually they are developed for special types of the presented IDEs.

We consider the following fourth-order IDEs in the RKHS

$$u^{(iv)}(x) = F\left(x, u(x), u'(x), u''(x), u'''(x), Tu(x)\right), \ a \leq x \leq b, \tag{4.1}$$

subject to the BC's:

$$case\ I: \ u(a) = \alpha_0,\ u'(a) = \alpha_1,\ u(b) = \beta_0,\ u'(b) = \beta_1, \tag{4.2}$$

$$case\ II: u(a) = \alpha_0, u''(a) = \alpha_2,\ u(b) = \beta_0, u''(b) = \beta_2, \tag{4.3}$$

where a, b, α_i, β_i, $i = 0,1,2$ are real finite constants, $Tu(x) = \int_a^x h(x,t)\, u(t)\, dt$, $u(x)$ is a solution to be determined, $h(x,t)$ is continuous function, and F is a linear or nonlinear function depending on the problem discussed. We suppose that Equation (4.1) have a unique smooth solution.

4.2 Several Reproducing Kernel Spaces

In this section, several reproducing kernels are constructed in order to solve Equation (4.1) using RKHS method. First of all, we construct the space $W_2^5[a,b]$ in which every function satisfies the BC's (4.2) and then utilize the space $W_2^1[a,b]$. The inner product space $W_2^5[a,b]$ is defined as $W_2^5[a,b] = \{u(x) \mid u^{(i)}, i = 0,1,2,3,4$ are Abs.C real-valued functions on $[a,b]$, $u^{(5)} \in L^2[a,b]$, and $u(a) = u'(a) = u(b) = u'(b) = 0\}$. The inner

product and the norm in $W_2^5[a,b]$ are defined, respectively, by

$$\langle u,v\rangle_{W_2^5}=\sum_{i=0}^{2}u^{(i)}(a)\,v^{(i)}(a)+\sum_{i=0}^{1}u^{(i)}(b)\,v^{(i)}(b)+\int_a^b u^{(5)}(y)v^{(5)}(y)dy, \tag{4.4}$$

and $||u||_{W_2^5}=\sqrt{\langle u,u\rangle_{W_2^5}}$, where $u,v\in W_2^5[a,b]$.

The space $W_2^1[a,b]$ is defined as $W_2^1[a,b]=\{u(x)\mid u$ is Abs.C real-valued functions on $[a,b]$ and $u'\in L^2[a,b]\}$. The inner product and the norm in $W_2^1[a,b]$ are given, respectively, by $\langle u,v\rangle_{W_2^1}=\int_a^b(u'v'+uv)\,dy$ and $||u||_{W_2^1}=\sqrt{\langle u,u\rangle_{W_2^1}}$, where $u,v\in W_2^1[a,b]$. It is a complete reproducing kernel space and its reproducing kernel is given by

$$R_x(y)=\frac{1}{2\sinh(b-a)}\left[\cosh(x+y-b-a)+\cosh(|x-y|-b+a)\right].$$

From the definition of the reproducing kernel spaces $W_2^1[a,b]$ and $W_2^5[a,b]$, we get $W_2^1[a,b]\supset W_2^5[a,b]$, for any $u\in W_2^5[a,b]$, and $\|u\|_{W_2^1}\le\|u\|_{W_2^5}$.

The space $W_2^5[a,b]$ is called a RKHS if for each fixed $x\in[a,b]$ and any $u(y)\in W_2^5[a,b]$, there exist $K_x(y)\in W_2^5[a,b]$, $y\in[a,b]$ such that $\langle u(y),K_x(y)\rangle_{W_2^5}=u(x)$. Next theorem formulate the reproducing kernel $K_x(y)$.

Theorem 4.2.1 *The space $W_2^5[a,b]$ is a complete RKHS and the reproducing kernel $K_x(y)$ is given by:*

$$K_x(y)=\begin{cases}\sum_{i=0}^{9}p_i(x)y^i, & y\le x;\\ \sum_{i=0}^{9}q_i(x)y^i, & y>x.\end{cases} \tag{4.5}$$

Proof. *Firstly, we show that the space $W_2^5[a,b]$ is RKHS. Afterward, we find out the expression form of the reproducing kernel function $K_x(y)$ in the space $W_2^5[a,b]$.*

Through several integrations by parts for Equation (4.4), *we obtain*

$$\int_a^b u^{(5)}(y)\,K_x^{(5)}(y)\,dy=\sum_{i=0}^{4}(-1)^{4-i}u^{(i)}(y)\,K_x^{(9-i)}(y)\Big|_{y=a}^{y=b}+(-1)^3\int_a^b u(y)\,K_x^{(10)}(y)\,dy.$$

Hence,

$$\begin{aligned}\langle u(y), K_x(y)\rangle_{W_2^5} &= \sum_{i=0}^{2} u^{(i)}(a) K_x^{(i)}(a) + \sum_{i=0}^{1} u^{(i)}(b) K_x^{(i)}(b) \\ &+ \sum_{i=0}^{4} (-1)^{4-i} u^{(i)}(y) K_x^{(9-i)}(y)\Big|_{y=a}^{y=b} - \int_a^b u(y) K_x^{(10)}(y)\, dy.\end{aligned}$$

Since $K_x(y) \in W_2^5[a,b]$, *it follows that* $K_x^{(i)}(a) = 0$, $K_x^{(i)}(b) = 0$, $i = 0,1$. *Again, since* $u \in W_2^5[a,b]$, *one obtains* $u^{(i)}(a) = 0$, $u^{(i)}(b) = 0$, $i = 0,1$. *Thus, if* $K_x^{(2)}(a) - K_x^{(7)}(a) = 0$, $K_x^{(i)}(a) = 0$ *for* $i = 5,6$, *and* $K_x^{(i)}(b) = 0$ *for* $i = 5,6,7$, *then*

$$\langle u(y), K_x(y)\rangle_{W_2^5} = \int_a^b u(y)\left(-K_x^{(10)}(y)\right) dy.$$

Now, for each $x \in [a,b]$, *if* $K_x(y)$ *also satisfies* $-K_x^{(10)}(y) = \delta(x-y)$, *where* δ *is the dirac-delta function, then* $\langle u(y), K_x(y)\rangle_{W_2^5} = u(x)$. *Obviously,* $K_x(y)$ *is the reproducing kernel of the space* $W_2^5[a,b]$.

The characteristic equation of $-K_x^{(10)}(y) = \delta(y-x)$ *is* $\lambda^{10} = 0$, *and their characteristic values are* $\lambda = 0$ *with* 10 *multiple roots. So, let*

$$K_x(y) = \begin{cases} \sum_{i=0}^{9} p_i(x) y^i, & y \le x; \\ \sum_{i=0}^{9} q_i(x) y^i, & y > x. \end{cases}$$

On the other hand, let $K_x(y)$ *satisfies* $K_x^{(m)}(x+0) = K_x^{(m)}(x-0)$, $m = 0,1,...,8$. *Integrating* $-K_x^{(10)}(y) = \delta(x-y)$ *from* $x-\varepsilon$ *to* $x+\varepsilon$ *with respect to* y *and let* $\varepsilon \to 0$, *we have the jump degree of* $K_x^{(9)}(y)$ *at* $y = x$ *given by* $K_x^{(9)}(x-0) - K_x^{(9)}(x+0) = 1$. *Through the last descriptions the unknown coefficient of Equation* (4.5) *can be obtained. This completes the proof.*

Without loss of generality, the unique representation of the reproducing kernel $K_x(y)$ is obtained at $a = 0$ and $b = 1$ in IDE (4.1) subject to the BC's (4.2) and

is given by:

$$K_{1x}(y) = \frac{(-1+x)^2y^2}{362880}\left[\left(\begin{array}{l} 5x^3(8-15y)y + x^6(19-12y)y + y^7 + xy^6(-9+2y) \\ +3x^5y(-10+3y) + x^7y(-4+3y) + 5x^4y(1+6y)+ \\ 3x^2\left(30240 - 60480y + 30240y^2 + 12y^5 - 6y^6 + y^7\right) \end{array}\right)\right];$$

$$K_{2x}(y) = \frac{x^2(-1+y)^2}{362880}\left[\left(\begin{array}{l} 90720y^2 + 36x^5y^2 - 9x^6y(1+2y) + x^7(1+2y+3y^2) \\ +xy^2(-181440 + 40y + 5y^2 - 30y^3 + 19y^4 - 4y^5)+ \\ 3x^2y^2\left(30240 - 25y + 10y^2 + 3y^3 - 4y^4 + y^5\right) \end{array}\right)\right].$$

where $K_{1x}(y) = \sum_{i=0}^{9} p_i(x)\, y^i$ and $K_{2x}(y) = \sum_{i=0}^{9} q_i(x)\, y^i$. We mention here that the kernel $K_x(y)$ is symmetric, that is $K_{1x}(y) = K_{2y}(x)$ for each x and y.

4.3 The Solution of Case One

In this section, the existence of the solution and the iterative method of obtaining the solution of IDEs (4.1) and (4.2) are developed in the space $W_2^5[a,b]$.

To do this, we define a differential operator $L : W_2^5[a,b] \to W_2^1[a,b]$ such that $Lu = u^{(4)}(x)$. After homogenization of the BC's (4.2), the IDE (4.1) and (4.2) can be converted into the equivalent form as follows:

$$\begin{aligned} &Lu(x) = F(x, u, u', u'', u''', Tu),\ a \le x \le b; \\ &u(a) = u'(a) = u(b) = u'(b) = 0, \end{aligned} \tag{4.6}$$

where $u(x) \in W_2^5[a,b]$ and $F(x, u, u', u'', u''', Tu) \in W_2^1[a,b]$.

Theorem 4.3.1 *L is a bounded operator from the space $W_2^5[a,b]$ to the space $W_2^1[a,b]$.*

Proof. *By reproducing property of $K_x(y)$ and Schwarz inequality, we can get*

$$\left|u^{(i)}(x)\right| = \left|\langle\, u(y), K_x^{(i)}(y)\,\rangle_{W_2^5}\right| \le \|\, K_x^{(i)}(y)\,\|_{W_2^5} \|u(y)\|_{W_2^5}.$$

Since $K_x^{(i)}(y)$, $i = 1, 2, ..., 5$ is uniformly bounded about x and y, we have $\left|u^{(i)}(x)\right| \le M_i$

$\|u(y)\|_{W_2^5}$. *Hence,*

$$\|Lu(x)\|_{W_2^1}^2 = \left\|u^{(4)}(x)\right\|_{W_2^1}^2 = \int_a^b \left(\left(u^{(4)}(t)\right)^2 + \left(u^{(5)}(t)\right)^2\right) dt \leq M\,\|u(x)\|_{W_2^5}^2,$$

where $M = (b-a)\left(M_4^2 + M_5^2\right)$. *Therefore L is a bounded linear operator.*

Now, we construct an orthogonal function system of the space $W_2^5[a,b]$. For a fixed dense set $\{x_i\}_{i=1}^{\infty}$ of $[a,b]$, let $\varphi_i(x) = R_{x_i}(x)$. So, from the properties of $R_x(y)$, for every $u(x) \in W_2^1[a,b]$, it follows that $\langle u(x), \varphi_i(x)\rangle_{W_2^1} = \langle u(x), R_{x_i}(x)\rangle_{W_2^1} = u(x_i)$. Additionally, let $\psi_i(x) = L^*\varphi_i(x)$, where L^* is the adjoint operator of L. Obviously, $\psi_i(x) \in W_2^5[a,b]$. In terms of the properties of $K_x(y)$, one obtains

$$\langle u(x), \psi_i(x)\rangle_{W_2^5} = \langle u(x), L^*\varphi_i(x)\rangle_{W_2^5} = \langle Lu(x), \varphi_i(x)\rangle_{W_2^1} = Lu(x_i),\, i = 1, 2, \ldots$$

From the above assumption, it is clear that

$$\psi_i(x) = L^*\varphi_i(x) = \langle L^*\varphi_i(x), K_x(y)\rangle_{W_2^5} = \langle \varphi_i(x), L_y K_x(y)\rangle_{W_2^1} = L_y K_x(y)\big|_{y=x_i},$$

that is, $\psi_i(x)$ can be expressed in the form $\psi_i(x) = L_y K_x(y)\big|_{y=x_i}$. The subscript y by the operator L indicates that the operator L applies to the function of y.

Theorem 4.3.2 *Suppose that the inverse operator L^{-1} for Equation (4.6) exists. If $\{x_i\}_{i=1}^{\infty}$ is dense on $[a,b]$, then $\{\psi_i(x)\}_{i=1}^{\infty}$ is the complete function system of the space $W_2^5[a,b]$.*

Proof. *For each fixed $u(x) \in W_2^5[a,b]$, let $\langle u(x), \psi_i(x)\rangle_{W_2^5} = 0$, $i = 1, 2, \ldots$. Then,*

$$\langle u(x), \psi_i(x)\rangle_{W_2^5} = \langle u(x), L^*\varphi_i(x)\rangle_{W_2^5} = \langle Lu(x), \varphi_i(x)\rangle_{W_2^1} = Lu(x_i) = 0.$$

Since $\{x_i\}_{i=1}^{\infty}$ is dense on $[a,b]$, therefore, $Lu(x) = 0$. It follows that $u(x) \equiv 0$ from the existence of L^{-1}. So, the proof of the Theorem is complete.

The orthonormal function system $\{\overline{\psi}_i(x)\}_{i=1}^{\infty}$ of the space $W_2^5[a,b]$ can be derived from Gram-Schmidt orthogonalization process of $\{\psi_i(x)\}_{i=1}^{\infty}$ as follows:

$$\overline{\psi}_i(x) = \sum_{k=1}^{i} \beta_{ik}\psi_k(x)\,,\ \ \beta_{ii} > 0, \tag{4.7}$$

where β_{ik} are orthogonalization coefficients and are given by Equation (1.19).

The structure of the next two theorems are as follows: Firstly, we will give the representation of the exact solution of the IDE (4.1) and (4.2) in the space $W_2^5[a,b]$. Afterward, the convergence of approximate solution $u_n(x)$ to the analytic solution will be proved.

Theorem 4.3.3 *For each* $u(x) \in W_2^5[a,b]$, *the series* $\sum_{i=1}^{\infty}\langle u(x), \overline{\psi}_i(x)\rangle \overline{\psi}_i(x)$ *is convergent in sense of the norm of* $W_2^5[a,b]$ *.On the other hand, if* $\{x_i\}_{i=0}^{\infty}$ *is dense on* $[a,b]$ *and the solution of the IDE* (4.1) *and* (4.2) *is unique, then this solution satisfies the form:*

$$u(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik}\left[F\left(x_k, u(x_k), u'(x_k), u''(x_k), u'''(x_k), Tu(x_k)\right)\right]\overline{\psi}_i(x)\,. \tag{4.8}$$

Proof. *Applying Lemma* 3.2.1, *it is easy to see that* $\{\overline{\psi}_i(x)\}_{i=0}^{\infty}$ *is the complete orthonormal basis of* $W_2^5[a,b]$. *Thus,* $u(x)$ *can be expanded in the Fourier series as* $u(x) = \sum_{i=1}^{\infty}\langle u(x), \overline{\psi}_i(x)\rangle \overline{\psi}_i(x)$. *Moreover, the space* $W_2^5[a,b]$ *is the Hilbert space, then the series*

$$\sum_{i=1}^{\infty}\langle u(x), \overline{\psi}_i(x)\rangle \overline{\psi}_i(x)$$

is convergent in sense of the norm of $W_2^5[a,b]$ *.Since* $\langle v(x), \varphi(x)\rangle = v(x_i)$ *for each* $v(x) \in W_2^1[a,b]$. *Hence, we have*

$$\begin{aligned} u(x) &= \sum_{i=1}^{\infty}\langle u(x), \overline{\psi}_i(x)\rangle \overline{\psi}_i(x) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\langle u(x), \psi_k(x)\rangle \overline{\psi}_i(x) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\left\langle u\left(x\right), L^{*}\varphi_{k}\left(x\right)\right\rangle \overline{\psi}_{i}\left(x\right) \\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\left\langle Lu\left(x\right), \varphi_{k}\left(x\right)\right\rangle \overline{\psi}_{i}\left(x\right) \\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\left\langle F\left(x, u\left(x\right), u'\left(x\right), u''\left(x\right), u'''\left(x\right), Tu\left(x\right)\right), \varphi_{k}\left(x\right)\right\rangle \overline{\psi}_{i}\left(x\right) \\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}F\left(x_k, u\left(x_k\right), u'(x_k), u''(x_k), u'''(x_k), Tu\left(x_k\right)\right)\overline{\psi}_{i}\left(x\right).
\end{aligned}
$$

The proof of the theorem is complete.

Further, the approximate solution is obtained by

$$
u_n\left(x\right) = \sum_{i=1}^{n}\sum_{k=1}^{i}\beta_{ik}\left[F\left(x_k, u\left(x_k\right), u'(x_k), u''(x_k), u'''(x_k), Tu_{k-1}\left(x_k\right)\right)\right]\overline{\psi}_{i}\left(x\right). \qquad (4.9)
$$

Remark 4.3.1 *If the IDE* (4.1) *and* (4.1) *is linear, then the analytical and the approximate solutions can be obtained directly from Equations* (4.8) *and* (4.9), *respectively. If the IDE* (4.1) *and* (4.1) *is nonlinear, the approximation solution can be obtained using the following iterative method.*

According to Equation (4.8), the representation of the solution of IDE (4.1) and (4.2) can be denoted by

$$
u\left(x\right) = \sum_{i=1}^{\infty} A_i\overline{\psi}_{i}\left(x\right), \qquad (4.10)
$$

where

$$
A_i = \sum_{k=1}^{i}\beta_{ik}F\left(x_k, u_{k-1}\left(x_k\right), u'_{k-1}(x_k), u''_{k-1}\left(x_k\right), u'''_{k-1}(x_k), Tu_{k-1}\left(x_k\right)\right). \qquad (4.11)
$$

In fact, A_i, $i = 1, 2, ..$, are unknown, we will approximate A_i using known B_i. For a numerical computation, we define initial function $u_0(x_1)$, put $u_0^{(i)}\left(x_1\right) = u^{(i)}\left(x_1\right)$, $i = 0, 1, 2, 3$, and define the n-term approximation to $u\left(x\right)$ by

$$
u_n\left(x\right) = \sum_{i=1}^{n} B_i\overline{\psi}_{i}\left(x\right), \qquad (4.12)
$$

where the coefficients B_i , $i = 1, 2, .., n$, are given by

$$\begin{cases} B_1 = \beta_{11} F\left(x_1, u_0\left(x_1\right), u_0'(x_1), u_0''\left(x_1\right), u_0'''(x_1), Tu_0\left(x_1\right)\right), \\ u_1(x) = B_1 \overline{\psi}_1\left(x\right), \\ B_2 = \sum\limits_{k=1}^{2} \beta_{2k} F\left(x_k, u_{k-1}\left(x_k\right), u_{k-1}'(x_k), u_{k-1}''\left(x_k\right), u_{k-1}'''(x_k), Tu_{k-1}\left(x_k\right)\right), \\ u_2(x) = B_1 \overline{\psi}_1\left(x\right) + B_2 \overline{\psi}_2\left(x\right), \\ \cdots \\ B_n = \sum\limits_{k=1}^{n} \beta_{nk} F\left(x_k, u_{k-1}\left(x_k\right), u_{k-1}'(x_k), u_{k-1}''\left(x_k\right), u_{k-1}'''(x_k), Tu_{k-1}\left(x_k\right)\right). \end{cases} \quad (4.13)$$

In the iteration process of Equation (4.12), we can guarantee that the approximation $u_n(x)$ satisfies the BC's of IDE (4.1) and (4.2). Next, we proof $u_n(x)$ in the iterative formula (4.12) is converge to the exact solution $u(x)$ of IDE (4.1) and (4.2).

The Lemma 4.3.1 through Lemma 4.3.3 are collected for future use.

Lemma 4.3.1 *If* $u(x) \in W_2^5[a,b]$, *then there exists* $M_i > 0$ *such that* $\left|\left|u^{(i)}\right|\right|_{C^4} \leq M_i \left|\left|u\right|\right|_{W_2^5}$, $i = 0, 1, 2, 3, 4$.

Proof. *Using the expression of* $K_x(y)$, *one gets,* $\parallel K_x^{(i)}(y) \parallel_{W_2^5} \leq M_i$, $i = 0, 1, 2, 3, 4$, *then for any* $x, y \in [a,b]$, *we have*

$$\left|u^{(i)}(x)\right| = \left|\langle\, u(x), K_x^{(i)}(x)\,\rangle_{W_2^5}\right| \leq \left\|u\left(x\right)\right\|_{W_2^5} \parallel K_x^{(i)}(x) \parallel_{W_2^5} \leq M_i \left\|u\left(x\right)\right\|_{W_2^5}.$$

So, we have $\left|\left|u^{(i)}\left(x\right)\right|\right|_{C^4} \leq M_i\ \left|\left|u\left(x\right)\right|\right|_{W_2^5}$, $i = 0, 1, 2, 3, 4$.

Corollary 4.3.1 $u_n(x)$ *and* $u_n^{(i)}(x)$ *uniformly convergent to* $u(x)$ *and* $u^{(i)}(x)$, $i = 1, 2, 3, 4$.

Proof. *From Theorem 4.3.3 and Lemma 4.3.1, for* $x \in [a,b]$, *we have* $\mid u^{(i)}(x) - u_n^{(i)}(x) \mid \leq \parallel u^{(i)}\left(x\right) - u_n^{(i)}(x) \parallel_{C^4} \leq M_i \left\|u\left(x\right) - u_n(x)\right\|_{W_2^5} \rightarrow 0$, $i = 0, 1, 2, 3, 4$.

Obviously, $u_n(x_n) \to u(y)$ as $x_n \to y$, $n \to \infty$. Thus, by means of the continuation of $Tu(\cdot)$, it is obtained that $Tu_n(x_n) \to Tu_n(y)$ *as* $n \to \infty$. Hence, by the continuity of F, Theorem 4.3.3 and Corollary 4.3.1, we have $F(x_n, u_{n-1}(x_n), u'_{n-1}(x_n), u''_{n-1}(x_n), u'''_{n-1}(x_n), Tu_{n-1}(x_n)) \longrightarrow F(y, u(y), u'(y), u''(y), u'''(y), Tu(y))$ *as* $n \to \infty$.

Lemma 4.3.2 $Lu_n(x_j) = F\left(x_j, u_{j-1}(x_j), u'_{j-1}(x_j), u''_{j-1}(x_j), u'''_{j-1}(x_j), Tu_{j-1}(x_j)\right)$, $j \leq n$.

Proof. *The proof will be obtained by induction as follows. If* $j \leq n$, *then*

$$\begin{aligned} Lu_n(x_j) &= \sum_{i=1}^{n} B_i L\overline{\psi}_i(x_j) \\ &= \sum_{i=1}^{n} B_i \left\langle L\overline{\psi}_i(x), \phi_j(x) \right\rangle_{W_2^1} \\ &= \sum_{i=1}^{n} B_i \left\langle \overline{\psi}_i(x), L^*\phi_j(x) \right\rangle_{W_2^5} \\ &= \sum_{i=1}^{n} B_i \left\langle \overline{\psi}_i(x), \psi_j(x) \right\rangle_{W_2^5}. \end{aligned}$$

The orthogonality of $\left\{\overline{\psi}_i(x)\right\}_{i=1}^{\infty}$ *yields that*

$$\begin{aligned} \sum_{l=1}^{j} \beta_{jl} Lu_n(x_l) &= \sum_{i=1}^{n} B_i \left\langle \overline{\psi}_i(x), \sum_{l=1}^{j} \beta_{jl}\psi_l(x) \right\rangle_{W_2^5} \\ &= \sum_{i=1}^{n} B_i \left\langle \overline{\psi}_i(xf), \overline{\psi}_j(x) \right\rangle_{W_2^5} \\ &= B_j = \sum_{l=1}^{j} \beta_{jl} F\left(x_l, u_{l-1}(x_l), u'_{l-1}(x_l), u''_{l-1}(x_l), u'''_{l-1}(x_l), Tu_{l-1}(x_l)\right). \end{aligned}$$

Now, if $j = 1$, then $Lu_n(x_1) = F(x_1, u_0(x_1), u'_0(x_1), u''_0(x_1), u'''_0(x_1), Tu_0(x_1))$, and if $j = 2$, then

$$\begin{aligned} \beta_{21} Lu_n(x_1) + \beta_{22} Lu_n(x_2) &= \beta_{21} F(x_1, u_0(x_1), u'_0(x_1), u''_0(x_1), u'''_0(x_1), Tu_0(x_1)) + \\ &\quad \beta_{22} F(x_2, u_1(x_2), u'_1(x_2), u''_1(x_2), u'''_1(x_2), Tu_1(x_2)). \end{aligned}$$

So, $Lu_n(x_2) = F(x_2, u_1(x_2), u'_1(x_2), u''_1(x_2), u'''_1(x_2), Tu_1(x_2))$. By induction, it is

obvious $Lu_n(x_j) = F\left(x_j, u_{j-1}(x_j), u'_{j-1}(x_j), u''_{j-1}(x_j), u'''_{j-1}(x_j), Tu_{j-1}(x_j)\right)$.

Theorem 4.3.4 *If $\{x_i\}_{i=0}^{\infty}$ is dense on $[a,b]$, then $u_n(x)$ in the iterative formula (4.12) convergent to the exact solution $u(x)$ of IDE (4.1) and (4.2) in the space $W_2^5[a,b]$ and $u(x) = \sum_{i=1}^{\infty} A_i \overline{\psi}_i(x)$, where A_i is given by Equation (4.11).*

Proof. *First of all, we will prove the convergence of $u_n(x)$. By Equation (4.12), we have $u_{n+1}(x) = u_n(x) + B_{n+1}\overline{\psi}_{n+1}(x)$. From the orthogonality of $\left\{\overline{\psi}_i(x)\right\}_{i=1}^{\infty}$, it follows that*

$$\begin{aligned} ||u_{n+1}||^2_{W_2^5} &= ||u_n||^2_{W_2^5} + (B_{n+1})^2 \\ &= ||u_{n-1}||^2_{W_2^5} + (B_n)^2 + (B_{n+1})^2 = ... = ||u_0||^2_{W_2^5} + \sum_{i=1}^{n+1}(B_i)^2. \end{aligned}$$

Thus, from boundedness of $||u_n||_{W_2^5}$, we have $\sum_{i=1}^{\infty}(B_i)^2 < \infty$, that is, $\{B_i\}_{i=1}^{\infty} \in l^2$.

Since $(u_m - u_{m-1}) \perp (u_{m-1} - u_{m-2}) \perp ... \perp (u_{n+1} - u_n)$ it follows that if $m > n$, then

$$\begin{aligned} ||u_m(x) - u_n(x)||^2_{W_2^5} &= ||u_m(x) - u_{m-1}(x) + u_{m-1}(x) - ... + u_{n+1}(x) - u_n(x)||^2_{W_2^5} \\ &= ||u_m(x) - u_{m-1}(x)||^2_{W_2^5} + \ ... \ + ||u_{n+1}(x) - u_n(x)||^2_{W_2^5} \\ &= \sum_{i=n+1}^{m}(B_i)^2. \end{aligned}$$

Consequently, we have $||u_m(x) - u_n(x)||^2_{W_2^5} \to 0$ as $n, m \to \infty$ and $\sum_{i=n+1}^{m}(B_i)^2 \to 0$. Considering the completeness of $W_2^5[a,b]$, there exists $u(x) \in W_2^5[a,b]$ such that $u_n(x) \to u(x)$ as $n \to \infty$ in sense of the norm of $W_2^5[a,b]$.

Second, we will prove that $u(x)$ is the solutions of IDE (4.1) and (4.2). Since $\{x_i\}_{i=0}^{\infty}$ is dense on $[a,b]$, for any $x \in [a,b]$, there exists subsequence $\{x_{n_j}\}$, such that $x_{n_j} \to x$ as $j \to \infty$. From Lemma 4.3.2, we have

$$Lu\left(x_{n_j}\right) = F\left(x_{n_j}, u_{n_j-1}(x_k), u'_{n_j-1}(x_{n_j}), u''_{n_j-1}\left(x_{n_j}\right), u'''_{n_j-1}\left(x_{n_j}\right), Tu_{n_j-1}(x_k)\right).$$

Hence, let $j \to \infty$, *by Corollary* 4.3.1 *and the continuity of* F, *we have*

$$Lu(x) = F(x, u(x), u'(x), u''(x), u'''(x), Tu(x)).$$

That is, $u(x)$ *satisfies the Equation* (4.1). *Since* $\overline{\psi}_i(x) \in W_2^5[a,b]$, clearly, $u(x)$ *satisfies the BC* (4.2). *In other words,* $u(x)$ *is the solution of IDE* (4.1) and (4.2), *where* $u(x) = \sum_{i=1}^{\infty} A_i \overline{\psi}_i(x)$ *and* A_i *is given by Equation* (4.11). *This complete the proof.*

The approximate solution $u_n^N(x)$ can be obtained by taking finitely many terms in the series representation of $u_n(x)$ and

$$u_n^N(x) = \sum_{i=1}^{N} \sum_{k=1}^{i} \beta_{ik} F\left(x_k, u_{n-1}(x_k), u'_{n-1}(x_k), u''_{n-1}(x_k), u'''_{n-1}(x_k), Tu_{n-1}(x_k)\right) \overline{\psi}_i(x).$$

4.4 The solution of Case Two

First of all, we will construct the space $W_2^5[a,b]$ in which every function satisfies the BC's (4.3) such that $W_2^5[a,b] = \{u(x) \mid u^{(i)}, i = 0,1,2,3,4$ are Abs.C real-valued functions on $[a,b]$, $u^{(5)} \in L^2[a,b]$, and $u(a) = u''(a) = u(b) = u''(b) = 0\}$. The inner product and the norm are defined as (4.4). It is easy to show that $W_2^5[a,b]$ is a RKHS and the reproducing kernel $T_x(y)$ is given by

$$T_x(y) = \begin{cases} \sum_{i=0}^{9} c_i(x) y^i, & y \le x; \\ \sum_{i=0}^{9} d_i(x) y^i, & y > x. \end{cases}$$

For IDE (4.1) and (4.3), the unique representation of the reproducing kernel $T_x(y)$ at $a = 0$ and $b = 1$ is obtained such as:

$$T_{1x}\left(y\right)=\frac{1}{c}xy\left(\begin{array}{l}(362880\ \ (362884-725782y^{2}+362903y^{3}-12y^{6}+9y^{7}-2y^{8})-6x^{2}\\(43895295360-87800025610y^{2}+43910173465y^{3}-10160696y^{5}+\\5806148y^{6}-1088673y^{7}+6y^{8})+3x^{3}(43896746880-87820346930y^{2}\\+43950816165y^{3}-30482088y^{4}+4354668y^{6}-1088709y^{7}+14y^{8})-\\12x^{6}(362880-2177292y+2903074y^{2}-1088667y^{3}+12y^{6}-9y^{7}+\\2y^{8})+9x^{7}\left(-362884+725782y^{2}-362903y^{3}+12y^{6}-9y^{7}+2y^{8}\right)\\-2x^{8}(-362882y^{-1}+362880+18y^{2}-21y^{3}+12y^{6}-9y^{7}+2y^{8}))\end{array}\right);$$

where $T_{1x}\left(y\right)=\sum_{i=0}^{9}c_i\left(x\right)y^i$ and $c=263365240320$. By the symmetry of $T_x\left(y\right)$, we have $T_{1y}\left(x\right)=T_{2x}\left(y\right)=\sum\limits_{i=0}^{9}d_i\left(x\right)y^i$ for each x and y.

Now, we define a differential operator $L:W_2^5\left[a,b\right]\rightarrow W_2^1\left[a,b\right]$ such that $Lu=u^{(4)}\left(x\right)$. After homogenization of the BC's (4.3), the IDE (4.1) and (4.3) can be converted into the equivalent form as follows:

$$\begin{aligned}&Lu\left(x\right)=F\left(x,u,u',u'',u''',Tu\right),\ x\in\left[a,b\right];\\&u\left(a\right)=u''(a)=u\left(b\right)=u''(b)=0,\end{aligned}\tag{4.14}$$

where $u\left(x\right)\in W_2^5\left[a,b\right]$ and $F\left(x,u,u',u'',u''',Tu\right)\in W_2^1\left[a,b\right]$. Obviously, L is a bounded linear operator.

In the similar manner, the orthonormal function system can be constructed and Theorems 4.3.3 and 4.3.4 holds.

4.5 Numerical Experiment

In order to have a clear overview of our method, several examples with known exact solutions are studied to demonstrate the accuracy of the present method. Results obtained by the method are compared with the analytical solution of each example and are found to be in good agreement with each other.

Example 4.5.1 (Hashim, 2006) *Consider the linear IDE*

$$u^{(iv)}(x) = x(1+e^x) + 3e^x + u(x) - \int_0^x u(t)dt, \ 0 \leq x \leq 1; \tag{4.15}$$

subject to the BC's, case I :

$$u(0) = 1, \ u'(0) = 1, \ u(1) = 1+e, \ u'(1) = 2e. \tag{4.16}$$

Or to the BC's, case II :

$$u(0) = 1, \ u''(0) = 2, \ u(1) = 1+e, \ u''(1) = 3e. \tag{4.17}$$

The exact solution of the two cases is $u(x) = 1 + xe^x$.

Using RKHS method, taking $x_i = \frac{i-1}{n-1}$, $i = 1, 2, ..., n$, *the approximate solutions* $u_n(x)$ *are calculated by Equation* (4.9). *However, the numerical results of IDE* (4.15) *and* (4.16) *at some selected grid points for* $n = 26$, *using the reproducing kernel* $K_x(y)$ *on* $[0,1]$, *are given in Table* 4.1. *Furthermore, the numerical results of IDE* (4.15) *and* (4.17) *at the same grid points for* $n = 26$, *using the reproducing kernel* $T_x(y)$ *on* $[0,1]$, *are given in Table* 4.2.

Table 4.1: Numerical results for Example 4.5.1, case I.

x	*Analytical Solution*	*Numerical Solution*	*Absolute Error*	*Relative Error*
0.0	1	1	$0.0000E-00$	$0.0000E-00$
0.1	1.11051709	1.11051738	2.92588×10^{-7}	2.6347×10^{-7}
0.2	1.24428055	1.24428151	9.60746×10^{-7}	7.72129×10^{-7}
0.3	1.40495764	1.40495937	1.72985×10^{-6}	1.23125×10^{-6}
0.4	1.59672987	1.59673225	2.3705×10^{-6}	1.4846×10^{-6}
0.5	1.82436063	1.82436334	2.70232×10^{-6}	1.48124×10^{-6}
0.6	2.09327128	2.09327389	2.61357×10^{-6}	1.24856×10^{-6}
0.7	2.40962689	2.40962899	2.09267×10^{-6}	8.68463×10^{-7}
0.8	2.78043274	2.78043401	1.26349×10^{-6}	4.54423×10^{-7}
0.9	3.21364280	3.21364321	4.12929×10^{-7}	1.28492×10^{-7}
1	3.71828182	3.71828182	$0.0000E-00$	$0.0000E-00$

Table 4.2: Numerical results for Example 4.5.1, case II

x	*Analytical Solution*	*Numerical Solution*	*Absolute Error*	*Relative Error*
0.0	1	1	$0.0000E-00$	$0.0000E-00$
0.1	1.11051709	1.11049839	1.87024×10^{-5}	1.68412×10^{-5}
0.2	1.24428055	1.24424672	3.38334×10^{-5}	2.71912×10^{-5}
0.3	1.40495764	1.40491489	4.27476×10^{-5}	3.04262×10^{-5}
0.4	1.59672987	1.59668533	4.4551×10^{-5}	2.79014×10^{-5}
0.5	1.82436063	1.82432043	4.02012×10^{-5}	2.20358×10^{-5}
0.6	2.09327128	2.09323930	3.19778×10^{-5}	1.52765×10^{-5}
0.7	2.40962689	2.40960436	2.2537×10^{-5}	9.35292×10^{-6}
0.8	2.78043274	2.78041887	1.38644×10^{-5}	4.98643×10^{-6}
0.9	3.21364280	3.21363624	6.56487×10^{-6}	2.04281×10^{-6}
1	3.71828182	3.71828182	$0.0000E-00$	$0.0000E-00$

Example 4.5.2 (Hashim, 2006) *Consider the nonlinear IDE*

$$u^{(iv)}(x) = 1 + \int_0^x e^{-t} u^2(t) dt, \ 0 < x < 1; \tag{4.18}$$

subject to the BC's, case I :

$$u(0) = 1, \ u'(0) = 1, \ u(1) = e, \ u'(1) = e. \tag{4.19}$$

Or to the BC's, case II :

$$u(0) = 1, \ u''(0) = 1, \ u(1) = e, \ u''(1) = e. \tag{4.20}$$

The exact solution of the two cases is $u(x) = e^x$.

Using RKHS method, taking $n = 10$ and $N = 1$ with $x_i = \frac{i}{n}$, $i = 0, 1, ..., n$. The numerical results of IDE (4.18) and (4.19) at some selected grid points, using the reproducing kernel $K_x(y)$ on $[0, 1]$, are given in Table 4.3.

Table 4.3: Numerical results for Example 4.5.2, case I.

x	*Analytical Solution*	*Numerical Solution*	*Absolute Error*	*Relative Error*
0.0	1	1	$0.0000E-00$	$0.0000E-00$
0.1	1.10517091	1.10517103	1.14811×10^{-7}	1.03885×10^{-7}
0.2	1.22140275	1.22140264	1.12511×10^{-7}	9.21163×10^{-8}
0.3	1.34985880	1.34985816	6.45214×10^{-7}	4.77986×10^{-7}
0.4	1.49182469	1.49182340	1.28963×10^{-6}	8.64466×10^{-7}
0.5	1.64872127	1.64871945	1.81841×10^{-6}	1.10292×10^{-6}
0.6	1.82211880	1.82211677	2.0262×10^{-6}	1.112×10^{-6}
0.7	2.01375270	2.01375091	1.79729×10^{-6}	8.92507×10^{-7}
0.8	2.22554092	2.22553975	1.17327×10^{-6}	5.27185×10^{-7}
0.9	2.45960311	2.45960270	4.08001×10^{-7}	1.65881×10^{-7}
1	2.71828182	2.71828182	$0.0000E-00$	$0.0000E-00$

Furthermore, the numerical results of IDE (4.18) and (4.20) at the same grid points, using the reproducing kernel $T_x(y)$ on $[0,1]$, are given in Table 4.4. Next, the numerical results of IDE (4.18), subject to (4.19), which include the absolute error at some selected nodes in $[0,1]$ for $u^{(m)}(x)$, $m = 0, 1, ..., 4$ are given in Table 4.5.

Table 4.4: Numerical results for Example 4.5.2, case II.

x	*Analytical Solution*	*Numerical Solution*	*Absolute Error*	*Relative Error*
0.0	1	1	$0.0000E-00$	$0.0000E-00$
0.1	1.10517091	1.10512941	415075×10^{-5}	375575×10^{-5}
0.2	1.22140275	1.22132182	809282×10^{-5}	662584×10^{-5}
0.3	1.34985880	1.34974418	11462×10^{-4}	849124×10^{-5}
0.4	1.49182469	1.49168565	13904×10^{-4}	932015×10^{-5}
0.5	1.64872127	1.64857014	151122×10^{-4}	916601×10^{-5}
0.6	1.82211880	1.82197016	148633×10^{-4}	815715×10^{-5}
0.7	2.01375270	2.01362212	130582×10^{-4}	648452×10^{-5}
0.8	2.22554092	2.2254433	976267×10^{-5}	438665×10^{-5}
0.9	2.45960311	2.45955067	524392×10^{-5}	213201×10^{-5}
1	2.71828182	2.71828182	$0.0000E-00$	$0.0000E-00$

Table 4.5: Absolute error for $u^{(m)}(x)$.

m	$x = 0.16$	$x = 0.48$	$x = 0.64$	$x = 0.96$
0	2.38892×10^{-8}	1.73257×10^{-6}	1.98879×10^{-6}	7.77328×10^{-8}
1	2.70497×10^{-6}	4.59697×10^{-6}	1.84908×10^{-6}	3.67133×10^{-6}
2	3.72451×10^{-5}	2.91777×10^{-5}	4.55729×10^{-5}	7.59487×10^{-5}
3	1.38986×10^{-4}	2.07264×10^{-4}	2.27562×10^{-5}	7.65190×10^{-4}
4	1.21680×10^{-3}	8.33908×10^{-4}	1.13022×10^{-3}	1.52652×10^{-3}

Chapter Five

Summary and Concluding Remarks

It is well known that the Hilbert Space theory is the foundation of modern mathematics. The reproducing kernel space $W_2^m[a,b]$ is a Hilbert Space with some special properties. So, all the properties of the Hilbert Space will be hold. Further, the space $W_2^m[a,b]$ possesses some special and better properties which could make some problems be solved easier. For instance, many problems studied in $L_2[a,b]$, which is a complete Hilbert space, requires large amount of integral computations, and such computations may be very difficult in some cases. Thus, the numerical integrals have to be calculated in the cost of losing some accuracy. However, the properties of the space $W_2^m[a,b]$ require no more integral computation for some functions, instead of computing some values of a function at some nodes. This simplification of integral computation not only improves the computational speed, but also improves the computational accuracy. It's widely application and better vision could be expected.

5.1 Conclusions

Most IDEs are usually difficult to solve analytically. In many cases, it is required to obtain the approximate solutions. For this purpose, we construct a novel RKHS and give the way to express reproducing kernel function $K_x(y)$. Further, we employed a reproducing kernel and its conjugate operator to construct the complete orthonormal basis in the space $W_2^m[a,b]$, by adding the initial and BC's to the space $W_2^m[a,b]$.

The main characteristic feature of RKHS method is that global approximation can be established on the whole solution domain and the convergence is uniform. Based on these, a new numerical algorithm is presented for solving a class of IDEs in the space $W_2^m[a,b]$, and the solution is given analytically. The analytical solution $u(x)$ and approximate solution $u_n(x)$ are represented in the form of series in the space

$W_2^m[a,b]$. Moreover, the approximate solution and its derivatives converge uniformly to the exact solution and its derivatives, respectively. Meanwhile, the error of the approximate solution is monotone decreasing in the sense of the norm of $W_2^m[a,b]$. The experimental results show that only a few number of iteration steps can be used for numerical purpose with a high degree of accuracy. Therefore, the present method is an accurate and reliable analytical technique for IDEs of first, second and fourth-order.

5.2 Future Recommendations

By constructing a new form of the reproducing kernel function and different RKHS to deal with the initial and boundary conditions, the present method employed can be generalized to solve other appropriate IDEs subject to the initial or the BC's. This is a smart technique to deal with the non-classical conditions.

In the near future, we intend to make more researches and more studies as a continuation to this work, we try to solve mixed Fredholm-Volterra IDEs. We mention some of these researches: application of RKHS method to higher IDEs, system of second-order IDEs and mixed integro-differential-difference equations.

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الحل العددي للمعادلات التفاضلية التكاملية باستخدام طريقة استنساخ نواة فضاء هيلبرت

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المشرف المشارك

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في هذه الرسالة, نقوم بدراسة نظرية استنساخ نواة فضاء هيلبرت المرتبطة بالنواة المعرفة الموجبة وتطبيقها على نوع من المعادلات التفاضلية التكاملية. كما ونقدم طريقة حاسوبية لإيجاد اقتران استنساخ نواة فضاء هيلبرت والمشغل المرافق لبناء نظام تعامد أحادي كامل في فضاء سوبوليف.

نطبق طريقة استنساخ نواة فضاء هيلبرت للحصول على حلول تقريبية للمعادلات التفاضلية التكاملية من الدرجة الأولى والثانية والرابعة, حيث يمثل الحل التحليلي والتقريبي على شكل سلسلة في فضاء سوبوليف. كما نوضح أن الحل التقريبي يتقارب بشكل منتظم من الحل المضبوط ونعرض طريقة تكرارية لحل فئة من المعادلات التفاضلية التكاملية غير الخطية من الرتبة الأولى والثانية والرابعة.

تم تقديم وحل أمثلة عددية متنوعة على هذا النوع من المعادلات التفاضلية التكاملية للدلالة على الكفاءة الحاسوبية والدقة العالية لطريقة استنساخ نواة فضاء هيلبرت.